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Dear Sir,

I am pleased to inform you that your paper entitled “**Regularity in Semigroups of Order-Preserving Full Contraction Mappings**” has been accepted for publication in the Nigerian Journal of Mathematics and Applications, volume 33, 2023 (ISSN: 0795-2767) subject to minor corrections as pointed out by our reviewers.

Congratulations and best wishes.

Prof. K. Rauf, Ph.D.

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Regularity in Semigroups of Order-Preserving Full Contraction Mappings

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ABSTRACT

A regular Semigroup is a semigroup S in which every element is regular, that is, for each element a , there exists an element x such that $axa = a$, while a Contraction map is $|x\alpha - y\alpha| \leq |x - y|$. Let $\mathcal{T}_n$ be the full symmetric semigroup on $X_n = \{1, 2, \dots, n\}$. Let also $\mathcal{O}_n = \{\alpha \in \mathcal{T}_n : (\forall x, y \in X_n) x \leq y \implies x\alpha \leq y\alpha \text{ or } x\alpha \geq y\alpha\}$ and $\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n : (\forall x, y \in X_n) |x - y| \geq |x\alpha - y\alpha|\}$. Then $\mathcal{CT}_n$ and $\mathcal{OCT}_n$ be its subsemigroups of contraction and order-preserving full contractions of X_n . This paper investigates the regularity of the subsemigroup, $\mathcal{OCT}_n$ and found that $\mathcal{OCT}_n$ is not a regular semigroup.

1. INTRODUCTION

The full transformation semigroup $\mathcal{T}_X$ on a set X , the semigroup analogue of the symmetric group S_X , has been much studied over the last seventy years, in both the finite and the infinite cases, please see [2, 3, 4, 7, 8] but the contraction semigroup cases have just come onboard in the last ten years, see also [1, 5, 6, 9]. Regular semigroups were introduced by Green's, [2], in his influential 1951 paper "On the structure of semigroups"; this was also the paper in which the popular Green's relations were introduced. The concept of regularity in a semigroup was adopted from an analogous condition for rings. It was his study of regular semigroups which led Green to define his celebrated the five Greens equivalent relations.

Let $X_n = \{1, 2, \dots, n\}$. Then a (partial) transformation $\alpha: \text{Dom } \alpha \subseteq X_n \rightarrow \text{Im } \alpha$ is said to be full if $\text{Dom } \alpha = X_n$; otherwise it is called strictly partial and a mapping in $\mathcal{T}_n$ is called *order preserving* if, for all x, y in $\{1, 2, \dots, n\}$,

$$x \leq y \implies x\alpha \leq y\alpha$$

and a *contraction* if

$$|x\alpha - y\alpha| \leq |x - y|$$

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for all x, y in $\{1, 2, \dots, n\}$. For undefined terms in semigroup theory, see [3, 4].

The methods employed in carrying out this research work were that the elements in the subsemigroup $\mathcal{OCT}_n$ were listed. The elements by their heights were arranged, within each height-class by their ranks as well as their images-type and kernel-type. From these arrangements, $\min$ and $\max$ were formed, the patterns of the arrangement were studied and results were deduced in each case through the principle of regularity definition.

2. PRELIMINARIES

This research work breaks into four sections. In the first section, the background to the study and a general introduction is given as well as the research methodology used in carrying out the research work were discussed, while section 2 deals with some preliminary results and algebraic structures of $\mathcal{OCT}_n$ were studied, also in section 3, regularity and research results were intensively investigated and presented, and finally, section 4, a general summary and conclusion were drawn.

Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}, \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_p$, $A_1 < A_2 < \cdots < A_p$, $a_i < a_{i-1} \leq \min A_i - \max A_{i-1}$, $b_1 < b_2 < \cdots < b_p$, $B_1 < B_2 < \cdots < B_p$, $b_i < b_{i-1} \leq \min B_i - \max B_{i-1}$, $i = 2, \dots, p$ ($p \geq 2$).

For any two elements in $\mathcal{OCT}_n$ with the same rank. Let

$d = \max A_1 - \max B_1$. If $\min A_p - \min B_p = d$ and

$$(2.1) \quad A_i = B_i + d \text{ or } B_i = A_i - d, i \in [2, p-1] \text{ (if } p \geq 3),$$

then α, β are called *same kernel-type* denoted by $\alpha \underline{Ker} \beta$.

Also let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}, \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix},$$

for any two elements in $\mathcal{OCT}_n$ with the same rank. If

$$(2.2) \quad |a_i - a_j| = |b_i - b_j| \text{ for all } i, j \in [1, p],$$

then α, β are called *same image-type* denoted $\alpha \underline{Im} \beta$.

Let

$$(2.3) \quad \mathcal{O}_n = \{\alpha \in \mathcal{T}_n : (\forall x, y \in X_n) x \leq y \implies x\alpha \leq y\alpha \text{ or } x\alpha \geq y\alpha\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all *order-preserving full* transformations of X_n , and also, let

$$(2.4) \quad \mathcal{CT}_n = \{\alpha \in \mathcal{T}_n : (\forall x, y \in X_n) |x - y| \geq |x\alpha - y\alpha|\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all *full contractions* of X_n .

Also, two functions are $\mathcal{L}$ -related if and only if they have the same image, $\mathcal{R}$ -related if and only if for x and y in the set X_n ; such functions are in the same table row(kernel). Similarly, two functions are $\mathcal{D}$ -related if and only if their image sets have the same size. Then we have the following lemmas:

Lemma 2.1. *The set of all contraction mappings $\mathcal{CT}_n$ of an n -chain is a semigroup.*

Proof. We are to show that the set of $\mathcal{CT}_n$ is (i) Closed (ii) Associative.

(i) Closure: Let $\alpha, \beta \in \mathcal{CT}_n$. Then $|x - y| \geq |x\alpha - y\alpha|$ and $|x - y| \geq |x\beta - y\beta|$. [Goal: to

show that $|x - y| \geq |x(\alpha\beta) - y(\alpha\beta)|$.

Now

$$\begin{aligned} |x(\alpha\beta) - y(\alpha\beta)| &= |(x\alpha)\beta - (y\alpha)\beta| \\ &\leq |(x\alpha) - (y\alpha)| \\ &\leq |x - y| \end{aligned}$$

Hence, $\alpha\beta \in \mathcal{CT}_n$.

(ii) Associativity and identity are inherited. $\square$

Lemma 2.2. *Let $\alpha \in \mathcal{CT}_n$ be such that $f(\alpha) = m$. Then $F(\alpha) = \{i, i+1, \dots, i+m-1\}$. Equivalently, $F(\alpha)$ is convex.*

Proof. Observe that it is sufficient to show that any point between two fixed points (of $\alpha \in \mathcal{CT}_n$) must also be a fixed point. Let $x, y \in F(\alpha)$. Then $x\alpha = x$ and $y\alpha = y$. Suppose also without loss of generality $x < x' < y$ for some $x' \in X_n$. Note that if $x'\alpha = x$, there is nothing to prove. Thus we consider two cases: (i) $x' > x'\alpha$; (ii) $x' < x'\alpha$. In the former, we have

$$|y - x'| = |y\alpha - x'| = |y\alpha - x' < y\alpha - x'\alpha| = |y\alpha - x'\alpha|$$

which implies that α is not a contraction. Hence we get a contradiction. In the latter, we have

$$|x' - x| = |x' - x\alpha| = |x' - x\alpha < x'\alpha - x\alpha| = |x'\alpha - x\alpha|$$

which implies that α is not a contraction. Hence we get a contradiction. Thus, the proof is complete. $\square$

Lemma 2.3. *Let $\alpha \in \mathcal{CT}_n$ be such that $h(\alpha) = p$. Then $im\alpha = \{i, i+1, \dots, i+p-1\}$. Equivalently, $im\alpha$ is convex.*

Proof. (By contradiction) Suppose that $im\alpha$ is not convex. Then there exist $x, z \in im\alpha$ with $x < y < z$ for some $y \in X_n \setminus im\alpha$. Let $(y-1]$ and $[y+1)$ the lower and upper saturations of $y-1$ and $y+1$, respectively. Notice that $x \in (y-1]$ and $z \in [y+1)$. Moreover, $(y-1]\alpha^{-1} \neq X_n \neq [y+1)\alpha^{-1}$ but $(y-1]\alpha^{-1} \cup [y+1)\alpha^{-1} = X_n$. If $(y-1]\alpha^{-1}$ is convex then since $(y-1]\alpha^{-1} \neq X_n$ there exist either

(i) $t \in (y-1]\alpha^{-1}$ and $t+1 \in [y+1)\alpha^{-1}$; or

(ii) $t \in (y-1]\alpha^{-1}$ and $t-1 \in [y+1)\alpha^{-1}$.

Case (i): it is clear that $t\alpha \leq y-1$ and $(t+1)\alpha \geq y+1$ so that

$$2 \leq (t+1)\alpha - t\alpha = |(t+1)\alpha - t\alpha| \leq |(t+1) - t| = 1,$$

which is a contradiction. Case (ii): it is clear that $t\alpha \leq y-1$ and $(t-1)\alpha \geq y+1$ so that

$$2 \leq (t-1)\alpha - t\alpha = |(t-1)\alpha - t\alpha| \leq |(t-1) - t| = 1,$$

which is another contradiction. On the other hand if $(y-1]\alpha^{-1}$ is not convex then there exists $t \in (y-1]\alpha^{-1}$ and either $t+1 \in [y+1)\alpha^{-1}$ or $t-1 \in [y+1)\alpha^{-1}$. In the former, we see that $t\alpha \leq y-1$ and $(t+1)\alpha \geq y+1$. Thus,

$$2 \leq (t+1)\alpha - t\alpha = |(t+1)\alpha - t\alpha| \leq |(t+1) - t| = 1,$$

which is a contradiction. In the latter, we see that $t\alpha \leq y-1$ and $(t-1)\alpha \geq y+1$. Thus,

$$2 \leq (t-1)\alpha - t\alpha = |(t-1)\alpha - t\alpha| \leq |(t-1) - t| = 1,$$

which is another contradiction. Hence, the proof is complete. $\square$

Lemma 2.4. Let $\alpha, \beta \in \mathcal{OCT}_n$. If $h(\alpha) = h(\beta)$ then α and β have the same image type ($\alpha \underline{Im} \beta$).

Proof. Since $Im \alpha$ is convex for all $\alpha \in \mathcal{OCT}_n$, by Lemma 2.3 it follows that $|a_i - a_{i+1}| = 1$ for all $i \in \{1, 2, \dots, p-1\}$. Hence, the proof is complete. $\square$

Lemma 2.5. Let $\alpha, \beta \in \mathcal{OCT}_n$ be such that $h(\alpha) = 1 = h(\beta)$. Then α and β have the same kernel type ($\alpha \underline{Ker} \beta$).

3. REGULAR SEMIGROUPS

A regular semigroup is a semigroup S in which every element is regular, i.e., for each element a , there exists an element x such that $axa = a$ and are one of the most-studied classes of semigroups, and their structure is particularly amenable to study via Green's relations.

Some years back, [9] initiated the algebraic study of semigroups of Order-preserving partial Contractions of an n -chain, where they referred to *Contractions* as *Compressions*.

Lemma 3.1. Let $\alpha \in \mathcal{CT}_n$ be such that $h(\alpha) = p \geq 3$. Then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ a_1 & a_2 & \dots & a_p \end{pmatrix}$$

is regular if and only if $LT(\alpha)$ or $UT(\alpha)$ (Lower or Upper Transversal) is convex.

Proof. Suppose $LT(\alpha)$ is convex. Without loose of generality : $\{i, i+1, \dots, p+i-1\} = LT(\alpha)$ where $1 \leq i \leq n-p+1$ and observe that $i+j-1 \in A_j$. Define α' by

$$a_j \alpha' = 1 + j - 1 (j = 1, 2, \dots, p)$$

$$(x \neq a_j) x \alpha' = \begin{cases} i, & \text{if } x < a_1; \\ p+i-1, & \text{if } x > a_1. \end{cases}$$

Also for $\alpha \alpha' \alpha = \alpha$

$$\begin{aligned} (y\alpha) \alpha' \alpha &= (a_j \alpha') \alpha \\ &= (i+j-1) \alpha \\ &= A_j \alpha \\ &= y \alpha. \end{aligned}$$

$\square$

In the following theorem, we give a characterisation of regular element in $\mathcal{OCT}_n = \mathcal{O}_n \cap \mathcal{CT}_n$.

Theorem 3.2. Let $\alpha \in \mathcal{O}_n$ and $\mathcal{CT}_n$ be as defined in equations (2.3) and (2.4) respectively and let $\alpha \in \mathcal{OCT}_n = \mathcal{O}_n \cap \mathcal{CT}_n$.

(i) If $h(\alpha) \leq 2$ then α is regular.

(ii) If $h(\alpha) > 2$ then α is regular if and only if $\min A_p - a_p = d$ and $A_i = \{a_i + d\}$.

Proof. Let (i) If $h(\alpha) = 1$ then α is clearly regular by [9, Theorem 2.2(1), p.1016].

Note that

$$\alpha = \begin{pmatrix} A_1 \\ a_1 \end{pmatrix}.$$

Let $x \in A_1$. Define β by

$$\beta = \begin{pmatrix} a_1 \\ x \end{pmatrix}.$$

Then $\beta \in \mathcal{OCT}_n$ and $\alpha = \alpha\beta\alpha$.

If $h(\alpha) = 2$ then $\alpha = \begin{pmatrix} A_1 & A_2 \\ a_1 & a_2 \end{pmatrix}$ where $\min A_2 - \max A_1 = 1$ by the convexity of A_1 and A_2 , and the fact that $A_1 \cup A_2 = X_n$. Moreover, $a_2 = a_1 + 1$ by Lemma 2.3. Thus

$$\min A_2 - \max A_1 = 1 = a_2 - a_1 \Rightarrow \max A_1 - a_1 = \min A_2 - a_2$$

and so α is regular by [9, Theorem 2.2(2), p.1017].

(ii) The direct half follows from [9, Theorem 2.2(3), p.1017]. If $h(\alpha) > 2$, then α is regular if and only if $\min A_p - a_p = d$ and $A_i = \{a_i + d\}$, $i \in [2, p-1]$.

Suppose that α is regular. Then there exists $\beta \in \mathcal{OCT}_n$ such that $\alpha = \alpha\beta\alpha$. Thus, for any $i, j \in [1, p]$, $a_i = A_i\alpha = (A_i\alpha)\beta\alpha = (a_i\beta)\alpha$ $a_j = A_j\alpha = (A_j\alpha)\beta\alpha = (a_j\beta)\alpha$, and so

$$(3.1) \quad a_i\beta \in A_i, \quad a_j\beta \in A_j.$$

Take any $i, j \in [1, p]$ with $i < j$. Note that $a_i < a_j$, $A_i < A_j$ and by equation (3.1). Since $\alpha, \beta \in \mathcal{OCT}_n$, we have

$$\begin{aligned} a_j - a_i &= |a_j - a_i| = |A_j\alpha - A_i\alpha| \\ &= |(\min A_j)\alpha - (\max A_i)\alpha| \\ &\leq |\min A_j - \max A_i| = \min A_j - \max A_i \\ &\leq a_j\beta - a_i\beta = |a_j\beta - a_i\beta| \\ &\leq |a_j - a_i| = a_j - a_i. \end{aligned}$$

Thus

$$a_j - a_i = \min A_j - \max A_i, \quad i < j, \quad i, j \in [1, p],$$

and so

$$(3.2) \quad \min A_j - a_j = \max A_i - a_i, \quad i < j, \quad i, j \in [1, p].$$

Clearly, $\min A_p - a_p = \max A_1 - a_1 = d$. By equation (3.2), we have

$$\min A_p - a_p = \max A_i - a_i, \quad \min A_i - a_i = \max A_1 - a_1, \quad i \in [2, p-1].$$

Thus $\max A_i - a_i = \min A_i - a_i (= d)$ and so $\max A_i = \min A_i = a_i + d$. It follows that $A_i = \{a_i + d\}$.

Conversely, suppose that $\min A_p - a_p = d$ and $A_i = \{a_i + d\}$ where $i \in [2, p-1]$. Define β by

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ \max A_1 & a_2 + d & \cdots & a_{p-1} + d & \min A_p \end{pmatrix}.$$

Then $\beta \in \mathcal{OCT}_n$ and $\alpha = \alpha\beta\alpha$. Define β by

$$\beta = \begin{cases} a_i\beta = \max A_i, & a_p\beta = \min A_p, \\ a_i\beta = \max A_i + d, & \text{if } (2 \leq i \leq p-1), \\ x\beta = a_i, & \text{if } x < a_1, \\ y\beta = a_p, & \text{if } x > a_p. \end{cases}$$

□

Example 3.3. For any two elements in OCT_n with the same rank (if $p \geq 3$). Then α, β are called *same kernel-type* denoted by $\alpha \underline{Ker} \beta$:

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \{3, 4, 5\} \\ 1 & 2 & 3 \end{pmatrix}, \beta = \begin{pmatrix} \{1, 2\} & \{3\} & \{4, 5\} \\ 1 & 2 & 3 \end{pmatrix}, \gamma = \begin{pmatrix} \{1, 2, 3\} & \{4\} & \{5\} \\ 1 & 2 & 3 \end{pmatrix}.$$

Solution. For $\alpha \underline{Ker} \beta$, we have $1 - 2 = -1 = d$ and $3 - 4 = -1 = d$, then

$$A_2 = \{3 + d\} = \{3 + (-1)\} = \{2\}$$

or

$$A_2 = \{2 - d\} = \{2 - (-1)\} = \{3\}$$

Also for $\beta \underline{Ker} \gamma$, we have

$2 - 3 = -1 = d = 4 - 5 = -1$, then

$$B_2 = \{4 + d\} = \{4 + (-1)\} = \{4 - 1\} = \{3\}$$

or

$$C_2 = \{3 - d\} = \{3 - (-1)\} = \{3 + 1\} = \{4\}.$$

Example 3.4. For any two elements in OCT_n with the same rank for all $i, j \in [1, p]$. Then α, β are called *same image-type* denoted $\alpha \underline{Im} \beta$.

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \{3, 4, 5\} \\ 1 & 2 & 3 \end{pmatrix}, \beta = \begin{pmatrix} \{1, 2\} & \{3\} & \{4, 5\} \\ 2 & 3 & 4 \end{pmatrix}, \gamma = \begin{pmatrix} \{1, 2, 3\} & \{4\} & \{5\} \\ 3 & 4 & 5 \end{pmatrix}.$$

Solution. For $\alpha \underline{Im} \beta$, where $|a_i - a_j| = |b_i - b_j|$ for all $i, j \in [1, p]$, we have

$$|1 - 2| = 1 \text{ and } |2 - 3| = 1,$$

also,

$$|2 - 3| = 1 \text{ and } |3 - 4| = 1.$$

Similarly, for $\beta \underline{Im} \gamma$, we have

$$|2 - 3| = 1 \text{ and } |3 - 4| = 1,$$

also,

$$|3 - 4| = 1 \text{ and } |4 - 5| = 1.$$

Theorem 3.5. Let $\alpha, \beta \in OCT_n$ be regular elements with $im \alpha = im \beta$ ($|im \alpha| \geq 2$). Then $\alpha \underline{Ker} \beta$.

Proof.

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}, \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_p, A_1 < A_2 < \cdots < A_p, B_1 < B_2 < \cdots < B_p$. Let $d_1 = \max A_1 - a_1$, $d_2 = \max B_1 - b_1$ and $d = \max A_1 - \max B_1$. Then $d = d_1 - d_2$.

Since α, β are regular, by Theorem 3.2, we have

$$\min A_p - a_p = \max A_1 - a_1 = d_1, \min B_p - b_p = \max B_1 - b_1 = d_2,$$

$$A_i = \{a_i + d_1\}, \quad B_i = \{a_i + d_2\}, \quad i \in [2, p-1] \text{ (if } p \geq 3).$$

It follows that

$$\begin{aligned} \min A_p - \min B_p &= \max A_1 - \max B_1 = d, \\ A_i &= \{a_i + d_2 + d\} = B_i + d, \quad i \in [2, p-1] \text{ (if } p \geq 3). \end{aligned}$$

Hence $\alpha \text{Ker} \beta \in \mathcal{OCT}_n$. □

Conclusions: In conclusion, the reasearch work investigated the algebraic structure of $\mathcal{OCT}_n$ just as the same as the results in [9]. With these results, we conclude that $\mathcal{OCT}_n$ is not regular for $n \geq 4$, that is, for each element a , there exists an element x such that $axa \neq a$. An example is given below.

Let

$$\alpha = \begin{pmatrix} \{1\} & \{2, 3\} & \{4\} \\ 1 & 2 & 3 \end{pmatrix} \in \mathcal{OCT}_n.$$

Then by Theorem 3.2(2), α is not regular in $\mathcal{OCT}_n$.

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REFERENCES

- [1] ADESHOLA A. D. & UMAR A. (2018). Combinatorial Results for Certain Semigroups of Order-Preserving Full Contraction Mappings of a Finite Chain. *Journal of Combinatorial Mathematics and Combinatorial Computing*. **106**, 37-49.
- [2] GREEN J. A. (1951). On the Structure of Semigroups. *Ann. of Math.* **54**, 163-172.
- [3] HOWIE J. M. (1995). *Fundamentals of semigroup theory*. Oxford: Clarendon Press.
- [4] HOWIE J. M. (2006). *Semigroups of Mappings: Technical report series*. King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.
- [5] IBRAHIM G. R., LAWAL O. J., AKINWUMI S. A., GATTA N. B. & ADESHOLA A. D. (2022). Some Combinatorial Results on Star-Like Transformation Semigroups. *Nigerian Journal of Mathematics and Applications*. **32**, 56-67.
- [6] IBRAHIM G. R., IMAM A. T., ADESHOLA A. D. & GATTA G. N. (2019). On some Algebraic Properties of Order-Preserving Full Contraction Transformation. *Journal of Semigroup Theory Appl.* **2019**:2.
- [7] PEI H. S. (2005). Regularity and Green's relations for semigroups of transformations that preserve an equivalence. *Comm. Algebra*. **33** (1), 109-118.
- [8] SUN L., PEI H. S. & CHENG Z. X. (2007). Regularity and Green's relations for semigroups of transformations preserving orientation and an equivalence. *Semigroup Forum*. **74**, (3), 473-486.
- [9] ZHAO P. & YANG M. (2012). Regularity and Green's relations on Semigroups of transformations preserving order and compression. *Bull Korean Math. Soc.* **49**, 1015-1025.