

APPROXIMATE SOLUTION OF NON-HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION VIA IMQ RADIAL BASIS FUNCTION

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Abstract

In recent years radial basis functions (RBFs) has play an important role in approximating functions, partial differential equations (PDEs). In this paper, the problem of solving non-homogeneous partial differential equations subject to the boundary conditions was considered. We employ Kansa's method to approximate the solution of PDEs using inverse multiquadric radial basis function (IMQRBf). The results of numerical experiments are presented, and compared with the exact solutions to confirm the effectiveness and the accuracy of the scheme. All the computation was carried out using MATLAB codes.

Keywords: Radial basis functions (RBFs), inverse multiquadric (IMQ), boundary conditions, collocation.

1. Introduction

Numerical method for solving PDE problems should be high-order accurate, flexible with respect to the computationally efficient, effectiveness and easy to implement. The most commonly used methods fulfill one or two of the criteria, but not all. Finite difference methods can be made high-order accurate, but require a structured grid (or a collection of well structured grids). Spectral methods are even more accurate, but have severe restrictions on the geometry. A fairly new approach for solving PDE problems is through radial basis functions (RBFs). The feature of RBFs is that it does not require a grid.

*In recent years, RBFs have gained more attention in different areas of scientific research such as interpolation problems [1-6], mesh free approximation methods for differential equations [7-10]. The numerical method which uses RBFs collocation method is a family of messless method [9]. And this has some advantages over mesh dependent methods such as boundary element methods, finite element methods, finite volume methods, finite-different methods and spectral methods. Due to the computational time involve providing suitable mesh to the domain of the problem in mesh-dependent methods. Messless methods has attracted more attention of researchers in science and engineering. In this paper, we consider RBF approximation for non-homogeneous partial differential equations via IMQ by using Kansa's method approach [1, 2] which was extended to solve different forms of differential equations [11- 15]

2. Radial Basis Function Interpolation

The RBF interpolation method uses linear combinations of N (N is the number of data points) radial function $\varphi(r)$, where $r = \|x - \tau_i\|$ and τ_i are the centers in $\square^d$, d is the dimension of the problem. The RBF interpolant takes the form

$$\Gamma_f(x) = \sum_{k=1}^N \beta_k \varphi(\|x - \tau_k\|, \varepsilon) \quad (2.1)$$

Where ε is the shape parameter and the coefficients β_k are determined by enforcing the interpolation condition.

$$\Gamma_f(x_j) = f(x_j) \quad j = 1, 2, 3, \dots, K \quad (2.2)$$

and when $N = K$, (2.2) becomes $N \times N$ linear system

$$A\beta = f \Rightarrow \beta = A^{-1}f \quad (2.3)$$

Where matrix A with entries

$$A = \begin{pmatrix} \varphi(\|x_1 - \tau_1\|) & \cdots & \varphi(\|x_1 - \tau_N\|) \\ \varphi(\|x_2 - \tau_1\|) & \cdots & \varphi(\|x_2 - \tau_N\|) \\ \vdots & \cdots & \vdots \\ \varphi(\|x_N - \tau_1\|) & \cdots & \varphi(\|x_N - \tau_N\|) \end{pmatrix} \quad (2.4)$$

is called the **Interpolation matrix**

When we are interested in evaluating the interpolant M points on set $\Pi := \{\lambda_j\}_{j=1}^M$, (2.1) becomes

$$\Gamma_f(\lambda_j) = \sum_{k=1}^N \beta_k \varphi(\|\lambda_j - \tau_k\|, \varepsilon), j=1, \dots, M \quad (2.5)$$

Here are some examples of RBF

Table 1: Commonly used RBF

Type of RBF	$\varphi(r, \varepsilon)$
Gaussian	$\exp(-\varepsilon^2 r^2)$
Inverse Multiquadric (IMQ)	$(1 + \varepsilon^2 r^2)^{-1/2}$
Multiquadric (MQ)	$(1 + \varepsilon^2 r^2)^{1/2}$
Wendland	$(1 - \varepsilon r)_+^4 (4r + 1)$
Thin Plate Spline (TPS)	$r^3 \log(r)$

3. Methodology

Consider

$$Lu(x, y) = f(x, y), (x, y) \in \Omega \quad (3.1)$$

Subject to the boundary conditions

$$Bu(x, a) = g(x), Bu(x, b) = g(x), Bu(a, y) = g(y) \text{ and } Bu(b, y) = g(y) \quad (3.2)$$

Where L is the differential operator and B is the boundary operator. Let Λ be set of N distinct centers that are divided into subsets with N_{int} centers, τ_{int} and other subset contains N_{Bdy} centers, τ_{Bdy} where boundary conditions are enforced. Using RBF collocation method, and apply the operator L to the RBF interpolant (2.1), we obtain

$$Lu(x_j) = \sum_{k=1}^N \beta_k L\varphi(\|x_j - \tau_k\|, \varepsilon), j=1, \dots, N_{int} \quad (3.3)$$

and

$$Bu(x_j) = \sum_{k=1}^N \beta_k B\varphi(\|x_j - \tau_k\|, \varepsilon), j = N_{int} + 1, \dots, N \quad (3.4)$$

From (3.1)-(3.4), we obtain $N \times N$ linear system of the form

$$A = \begin{pmatrix} L\varphi(\|x_1 - \tau_1\|) & \cdots & L\varphi(\|x_1 - \tau_N\|) \\ \vdots & \cdots & \vdots \\ L\varphi(\|x_{N_{int}} - \tau_1\|) & \cdots & L\varphi(\|x_{N_{int}} - \tau_N\|) \\ B\varphi(\|x_{N_{int}+1} - \tau_1\|) & \cdots & B\varphi(\|x_{N_{int}+1} - \tau_N\|) \\ \vdots & \cdots & \vdots \\ B\varphi(\|x_N - \tau_1\|) & \cdots & B\varphi(\|x_N - \tau_N\|) \end{pmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_{N_{int}}) \\ g(x_{N_{int}+1}) \\ \vdots \\ g(x_N) \end{bmatrix} \quad (3.5)$$

Assuming the system matrix is invertible, then one can obtain the approximate solution at any M points by using the coefficients β in (2.1)

4. Numerical Experiments

In this section, we implement IMQRBF interpolant on non-homogeneous PDE and The root mean square (RMS) errors were calculated by the formula

$$RMS = \sqrt{\frac{\sum_{x \in M} |f(x) - \Gamma_f(x)|^2}{M}}$$

Example 4.1

In this example, we consider Klein-Gordon equation

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} + u(x, y) = 2 \sin x, \quad (x, y) \in [0, 1] \times [0, 1] \quad (4.1a)$$

Subject to boundary conditions

$$u(x, 0) = \sin x, \quad u(0, y) = \sin y, \text{ and } u(1, y) = \sin 1 + \sin y, \quad y \in [0, 1] \quad (4.1b)$$

With analytical solution

$$u(x, y) = \sin x + \sin y \quad (4.1c)$$

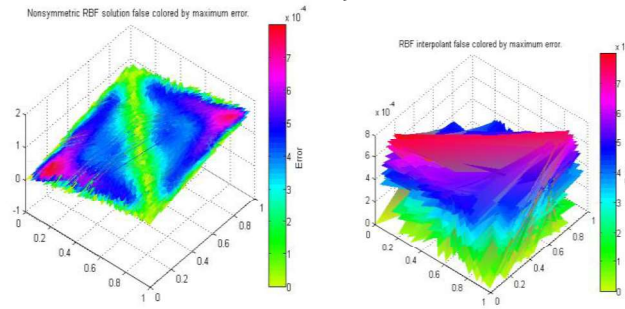


Figure 1: IMQRBF solution and the maximum error for $N = 50 \times 50$ and $\epsilon = 1$ for example 4.1

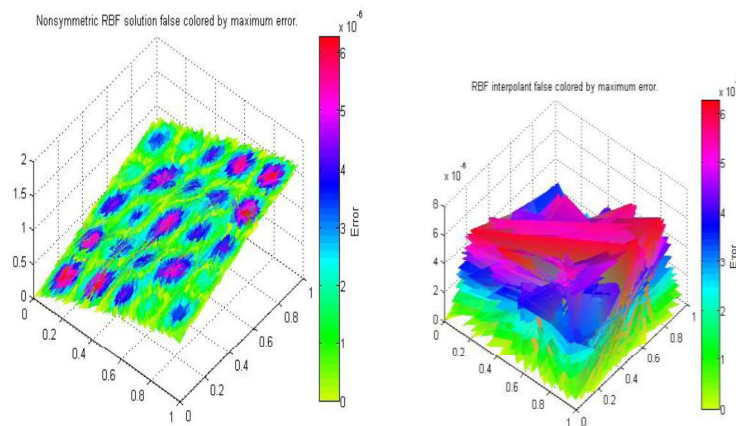


Figure 2: IMQRBF solution and the maximum error for $N = 289 \times 289$ and $\epsilon = 1$ for

Example 4.1Table 2: **IMQRBf using Non-symmetric collocation and $\epsilon = 1$ for example 4.1**

N	RMS
50 X 50	3.93×10^{-4}
150 X 150	2.95×10^{-6}
289 X 289	2.52×10^{-6}

Example 4.2

We consider example from [16]

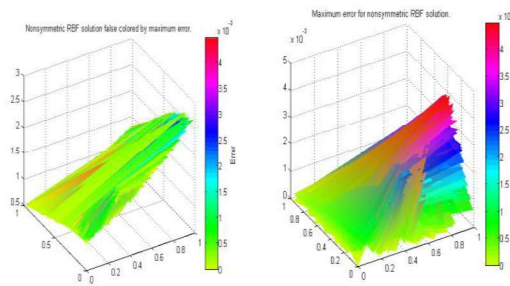
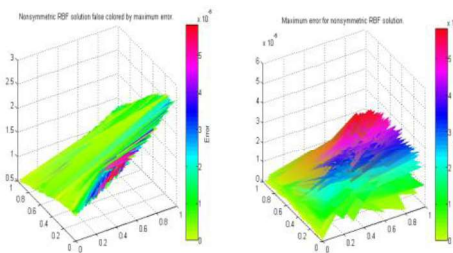
$$\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} = -\frac{(1+y)^2 \exp(x)}{(1+y^2)^2}, \quad (x, y) \in [0, 1]^2 \quad (4.2a)$$

With supplementary conditions

$$u(x, 0) = \exp(x), \quad u(1, y) = \frac{\exp(1)}{1+y^2}, \quad u(0, y) = \frac{1}{1+y^2} \quad (4.2b)$$

and the exact solution is

$$u(x, y) = \frac{\exp(x)}{1+y^2}$$

Figure 3: IMQRBf solution and the maximum error for N = 50 X 50 and $\epsilon = 1$ for Example 4.2Figure 4: IMQRBf solution and the maximum error for N = 289 X 289 and $\epsilon = 1$ for example 4.2Table 2: **IMQRBf using Non-symmetric collocation and $\epsilon = 1$ for example 4.2**

N	RMS
50 X 50	1.51×10^{-3}
150 X 150	2.99×10^{-5}
289 X 289	1.59×10^{-6}

Example 4.3

Consider

$$\alpha(x, y) \frac{\partial^2 u}{\partial x^2} - \beta(x) \frac{\partial u}{\partial x} - \gamma(x, y) \frac{\partial u}{\partial x} + \gamma(x, y) \frac{\partial^2 u}{\partial x^2} = g(x, y), \quad (x, y) \in [0, 1] \times [0, 1] \quad (4.3a)$$

where $\alpha(x, y) = 2 - x^2 - y^2$, $\beta(x) = 2x$, $\gamma(x, y) = \exp(x - y)$ and $g(x, y) = 32y(1 - y)(3x^2 + y^2 - x - 2) - 16x(1 - x)(3 - 2y)\exp(x - y)$

With boundary conditions

$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0 \quad (4.3b)$$

The exact solution is

$$u(x, y) = 16x(1 - x)y(1 - y) \quad (4.3c)$$

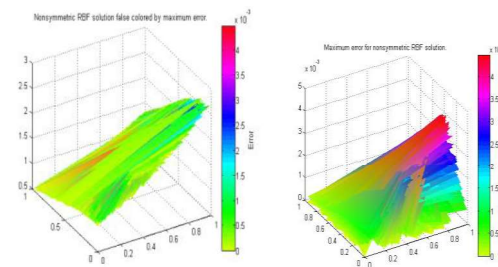


Figure 5: IMQRBF solution and the maximum error for $N = 50 \times 50$ and $\epsilon = 1$ for Example 4.3

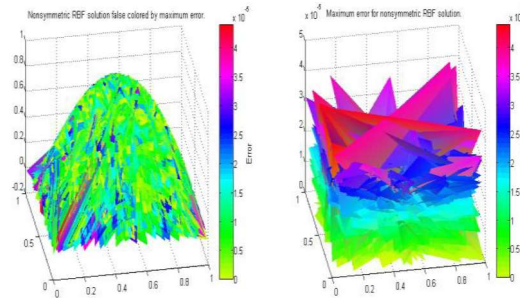


Figure 6: IMQRBF solution and the maximum error for $N = 289 \times 289$ and $\epsilon = 1$ for Example 4.3

Table 4: **IMQRBF** using Non-symmetric collocation and $\epsilon = 1$ for Example 4.3

N	RMS
50 X 50	7.18×10^{-3}
150 X 150	2.12×10^{-4}
289 X 289	1.28×10^{-5}

5. Discussion of Results and Conclusion

5.1. Discussion of Results

Table [2]-[4] are the RMS errors obtained from interpolating examples with shape parameter $\epsilon = 1$. The data sites used in this work are $N = [50, 150, 289]$. It is visually obvious from the tables that the errors follow a pattern irrespective of the PDE (i.e the RMS errors reduce as the value of the data sites increases) and change in the value of shape parameter may yield better accuracy depending on the optimal shape parameter, which is an open research.

Figure 1-6 are the interplants and the corresponding maximum errors for each examples

considered and the smoothness of these graphs also depend on the shape parameter, which has to be chosen wisely, as discussed in [17-19].

5.2. Conclusion

In this paper, we discussed 2D non-homogeneous linear PDE using Kansa's method approach by fixing shape parameter $\epsilon = 1 \forall N$ and this was applied to some selected problems and all the computation was done using MATLAB codes. The results obtained were tabulated in Table 2, 3 and 4, which confirm the accuracy, the effectiveness and simplicity of the method.

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