



Construction of ν -orthogonal polynomials and its implementation on differential equation

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ABSTRACT

In this paper, we construct some orthogonal polynomials using weight function of Jacobi polynomial $P_j^{(\alpha, \beta)}(x)$, $j \geq 0$ with $\alpha, \beta > -1$, by varying the values of α, β . The obtained polynomials were used respect to each weight functions to obtain the approximate solutions to an ordinary differential equations via recursive formulation analogue to the tau method. The effectiveness and accuracy of the scheme were tested on some selected problems and the significant of the proposed technique as it give the same degree of accuracy with results obtained using recursive tau method.

1. INTRODUCTION

In 1938, Lanczos [1] proposed the tau method for solving the linear differential equation, he then improved the method by proposing an operational technique for approximating non-linear ordinary differential equations (ODE) with some supplementary conditions based on the Tau method as discussed in [2, 3] since then considerable effort have been done to improve the accuracy and the effectiveness of the method. [4, 5] proposed the generalization of recursive tau method for

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solving linear and non-linear differential equations, [6] investigated the error estimate for both linear and non-linear ordinary differential equations, the canonical polynomials involved in the recursive formulation of the tau method was reported in [7] and its generalization was reported in [8], since its establishment, various researchers have extended the tau method to other form of differential equations, [9, 10, 11] employed tau method to estimate the error of Fredholm-Volterra integro-differential equations, [12] studied the effectiveness of some methods for solving higher order differential equations.

This paper is organized as follows: In section 2, we present the statement of the problem under consideration, in 2.2 we review some notable orthogonal polynomials. The description of the method is given in section 3 which includes construction of ν orthogonal polynomials reference to the weight functions under consideration and description of ν -recursive form is presented in section 3.2, numerical examples are presented in section 4 with concluding remarks given in section 4.

2. PRELIMINARIES

2.1. Statement of the problem. In this paper, the problem under consideration is ODE of the form:

$$(1) \quad Ly(x) \equiv \sum_{r=0}^m \left(\sum_{k=0}^{N_r} P_{rk} x^k \right) y^r(x) = \sum_{r=0}^{\sigma} f_r x^r, a \leq x \leq b$$

together with boundary conditions given by:

$$(2) \quad L^*y(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, k = 1, 2, \dots, m,$$

where $P_{rk}, f_r, a_{rk}, \alpha_k$ are constants, L is linear differential operator. We seek an approximation of the form:

$$(3) \quad y_n(x) = \sum_{r=0}^n \nu_r x^r,$$

where $\nu_r, r = 0, 1, \dots, N$ are the unknowns to be determined.

Equation (1) will be solved by transforming the equation into system of algebraic equations via recursive formulation analogue to the tau method using the derived orthogonal polynomials respect to the weight function generated from Jacobi polynomial.

2.2. Review of some notable orthogonal polynomials. $\chi_i(t)$ is said to be orthogonal reference to the weight function $\omega(t)$, $t \in [a, b]$, if the inner product of $\chi_i(t)$ satisfies the following:

$$(4) \quad \langle \chi_i(t), \chi_j(t) \rangle = \int_a^b \omega(t) \chi_i(t) \chi_j(t) dt = \begin{cases} 0, & i \neq j \\ \lambda_i, & i = j \end{cases}$$

Some of these polynomials $\chi_i(t)$ are:

2.2.1. Legendre polynomials. Polynomial $P_i(t)$, $t \in [-1, 1]$ is an orthogonal polynomial with respect to the weight function $\omega(t) = 1$ satisfying the recurrence relation:

$$(5) \quad P_{i+1}(t) = \frac{1}{i+1} [(2i+1)tP_i(t) - iP_{i-1}(t)], i \geq 1,$$

with $P_0(t) = 1, P_1(t) = t$.

See [13] for more details.

2.2.2. Laguerre polynomials. The generalized Laguerre polynomial $L_i^{(\beta)}(t)$ of degree i in β with respect to the weight function $\omega(t) = t^\beta \exp(-t)$, $\beta > -1$, $t \in (0, \infty)$ is defined as:

$$(6) \quad L_i^{(\beta)}(t) = \sum_{j=0}^i (-1)^j \binom{i+\beta}{i-j} \frac{t^j}{j!},$$

the first two of the polynomials are $L_0^{(\beta)}(t) = 1$, $L_1^{(\beta)}(t) = \beta+1-t$ with recurrence relation

$$(7) \quad L_{i+1}^{(\beta)}(t) = \frac{1}{i+1} [(\beta+2i+1-t)L_i^{(\beta)}(t) - (\beta+i)L_{i-1}^{(\beta)}(t)]$$

The most frequently used Laguerre polynomial is under the condition $\beta = 0$ is defined as:

$$(8) \quad L_i(t) = \sum_{j=0}^i (-1)^j \binom{i}{j} \frac{t^j}{j!},$$

with recurrence relation:

$$(9) \quad L_{i+1}(t) = \frac{1}{i+1} [(2i+1-t)L_i(t) - iL_{i-1}(t)].$$

The inner product is given by:

$$(10) \quad \langle L_i^{(\beta)}(t), L_j^{(\beta)}(t) \rangle = \int_0^\infty \omega(t) t^\beta \exp(-t) L_i^{(\beta)}(t) L_j^{(\beta)}(t) dt = \begin{cases} 0, & i \neq j \\ \frac{i! \Gamma(j+\beta+1)}{\Gamma(\beta+1)}, & i = j. \end{cases}$$

See [17] for more details.

2.2.3. *κ -th Kind Chebyshev polynomials.* κ -th kind Chebyshev polynomials are family of orthogonal polynomials with various kinds, the prominent ones are given here reference to their respective weight functions $\omega(t), t \in [-1, 1]$:

$$(11) \quad \chi_i = \begin{cases} T_i(t) = \cos(it), \omega(t) = \frac{1}{\sqrt{1-t^2}}, \kappa = 1 \\ U_i(t) = \frac{\sin(i+1)t}{\sin(t)}, \omega(t) = \sqrt{1-t^2}, \kappa = 2 \\ V_i(t) = \frac{\cos(i+\frac{1}{2})t}{\cos(\frac{t}{2})}, \omega(t) = \sqrt{\frac{1+t}{1-t}}, \kappa = 3 \\ W_i(t) = \frac{\sin(i+\frac{1}{2})t}{\sin(\frac{t}{2})}, \omega(t) = \sqrt{\frac{1-t}{1+t}}, \kappa = 4. \end{cases}$$

The inner product for the κ -th kind Chebyshev polynomials in the interval $[-1, 1]$ are:

$$(12) \quad \langle \chi_i(t), \chi_j(t) \rangle = \begin{cases} \langle T_i(t), T_j(t) \rangle = \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_i(t) T_j(t) dt = \begin{cases} 0, i \neq j \\ \frac{\pi}{2}, i = j, \end{cases} \\ \langle U_i(t), U_j(t) \rangle = \int_{-1}^1 \sqrt{1-t^2} U_i(t) U_j(t) dt = \begin{cases} 0, i \neq j \\ \frac{\pi}{2}, i = j \end{cases} \\ \langle V_i(t), V_j(t) \rangle = \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} V_i(t) V_j(t) dt = \begin{cases} 0, i \neq j \\ \pi, i = j \end{cases} \\ \langle W_i(t), W_j(t) \rangle = \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} W_i(t) W_j(t) dt = \begin{cases} 0, i \neq j \\ \pi, i = j \end{cases} \end{cases}.$$

See [16, 14, 15] for more details.

2.2.4. *Shifted Gegenbauer polynomials.* Gegenbauer polynomials $C_i^{(\alpha)}(t), t \in [-1, 1]$ with respect to the weight function $\omega(t) = (1-t^2)^{(\alpha-\frac{1}{2})}$ is defined as:

$$(13) \quad C_i^{(\alpha)}(t) = \sum_{j=0}^i \frac{(-1)^j \Gamma(2\alpha + 2i - j) \Gamma(\alpha + \frac{1}{2})}{(i-j)! \Gamma(2\alpha) \Gamma(j+1) \Gamma(i-j+\alpha + \frac{1}{2})} t^{i-j}.$$

with the recurrence relation:

$$(14) \quad C_{i+1}^{(\alpha)}(t) = \frac{1}{i+1} \left[2(i+\alpha)t C_i^{(\alpha)}(t) - (i+2\alpha-1) C_{i-2}^{(\alpha)}(t) \right], i \geq 1,$$

where $C_0^{(\alpha)}(t) = 1$, $C_1^{(\alpha)}(t) = 2\alpha t$. The corresponding shifted Gegenbauer polynomial $C_i^{(\alpha)*}(t)$, $t \in [a, b]$ is derived using the transformation $x = \frac{2t-(a+b)}{b-a}$ as:

$$(15) \quad C_{i+1}^{(\alpha)*}(t) = \frac{1}{i+1} \left[2(i+\alpha) \left(\frac{2t-(a+b)}{b-a} \right) C_i^{(\alpha)*}(t) - (i+2\alpha-1) C_{i-1}^{(\alpha)*}(t) \right], i \geq 1$$

where $C_0^{(\alpha)*}(t) = 1$, $C_1^{(\alpha)*}(t) = 2\alpha \left(\frac{2t-(a+b)}{b-a} \right)$

The analytic form corresponding to Eq. (15) in the interval $t \in [0, 1]$ is given as:

$$(16) \quad C_i^{(\alpha)*}(t) = \sum_{j=0}^i \frac{(-1)^j \Gamma(2\alpha + 2i - j) \Gamma(\alpha + \frac{1}{2})}{(i-j)! \Gamma(j+1) \Gamma(i-j+\alpha + \frac{1}{2}) \Gamma(2\alpha)} t^{i-j}.$$

with the following orthogonality condition:

$$(17) \quad \langle C_i^{(\alpha)*}(t), C_j^{(\alpha)*}(t) \rangle = \int_0^1 (t-t^2)^{(\alpha-\frac{1}{2})} C_i^{(\alpha)*}(t) C_j^{(\alpha)*}(t) dt = \begin{cases} 0, & \text{for } i \neq j \\ \frac{\pi 2^{1-4\alpha} \Gamma(j+2\alpha)}{j! [\Gamma(\alpha)]^2 (j+\alpha)}, & \text{for } i = j \end{cases}$$

See [20, 19] for more details.

2.2.5. Jacobi polynomials. The polynomials $P_i^{(\alpha, \beta)}(t)$, $i \geq 0$ with $\alpha, \beta > -1$ reference to the weight function $\omega(t) = (1-t)^\alpha (1+t)^\beta$ is defined as:

$$(18) \quad P_i^{(\alpha, \beta)}(t) = \frac{\Gamma(\alpha + i + 1)}{i! \Gamma(\alpha + \beta + i + 1)} \sum_{j=0}^i \frac{i! \Gamma(\alpha + \beta + i + j + 1)}{(i-j)! j! \Gamma(\alpha + j + 1)} \left(\frac{t-1}{2} \right)^j,$$

where $P_0^{(\alpha, \beta)}(t) = 1$, $P_1^{(\alpha, \beta)}(t) = (\alpha + 1) + (\alpha + \beta + 2) \left(\frac{t-1}{2} \right)$. When $\alpha = \beta = 0$, then it gives Legendre polynomials, $\alpha = \beta = \frac{1}{2}$ gives second kind Chebyshev polynomials and also when $\alpha = \beta \neq 0$ we have Gegenbauer polynomials of certain kinds.

See [18, 16] for details.

3. DESCRIPTION OF THE METHOD

In this section, we construct the ν orthogonal polynomials by varying the values of α and β in the weight function of Jacobi polynomial, then construct its equivalent shifted form. Subsequently, describe the methodology involve in the approximation of the differential equation using the derived orthogonal polynomials.

3.1. Construction of ν -orthogonal polynomials. Orthogonal polynomials have been widely used in the problems involving the approximation of functions such as in the economization of power series, Gaussian quadrature techniques, mini-max approximation, solution of both integral, differential equation as well as in collocation techniques and in recursive formulation of tau method among others.

The construction of orthogonal polynomials are based on three equations of Gram Schmidt orthogonalization principle namely:

$$(19) \quad \chi_j(x) = \sum_{i=0}^j C_i^{(j)} x^i,$$

$$(20) \quad \chi_j(1) = 1, \forall j \geq 0,$$

$$(21) \quad \langle \chi_i(x), \chi_j(x) \rangle = 0 \equiv \int_a^b \omega(x) \chi_i(x) \chi_j(x) dx, i \neq j.$$

where $\chi_j(x)$, $a \leq x \leq b$ is the orthogonal system in question.

To construct the ν orthogonal polynomials we employ Eqs. (19)-(21) using Jacobi weight function $\omega(x) = (1-x)^\alpha(1+x)^\beta$ at specific values of α and β , that's $(\alpha = 1, \beta = 0)$, $(\alpha = 0, \beta = 1)$ and $(\alpha = 1 = \beta = 1)$

For Case: $\alpha = 1, \beta = 0$, that's $\omega(x) = 1 - x, x \in [-1, 1]$

Employing Eqs. (19)-(21), we obtain the following polynomials:

$$\begin{aligned} \chi_0(x) &= 1, \chi_1(x) = \frac{1}{4}(1 + 3x), \chi_2(x) = \frac{1}{6}(-1 + 2x + 5x^2), \chi_3(x) = \frac{1}{32}(-3 - 15x + 15x^2 + 35x^3), \\ \chi_4(x) &= \frac{1}{40}(3 - 12x - 42x^2 + 28x^3 + 63x^4), \chi_5(x) = \frac{1}{96}(5 + 35x - 70x^2 - 210x^3 + 105x^4 + 231x^5), \\ \chi_6(x) &= \frac{1}{112}(-5 + 30x + 135x^2 - 180x^3 - 495x^4 + 198x^5 + 429x^6), \\ \chi_7(x) &= \frac{1}{1024}(-35 - 315x + 945x^2 + 3465x^3 - 3465x^4 - 9009x^5 + 3003x^6 + 6435x^7) \\ \chi_8(x) &= \frac{1}{1152}(35 - 280x - 1540x^2 + 3080x^3 + 10010x^4 - 8008x^5 - 20020x^6 + 5720x^7 + 12155x^8), \\ (22) \quad \chi_9(x) &= \frac{1}{2560}(63 + 693x - 2772x^2 - 12012x^3 + 18018x^4 + 54054x^5 - 36036x^6 - 87516x^7 \\ &\quad + 21879x^8 + 46189x^9) \end{aligned}$$

For Case: $\alpha = 0, \beta = 1$, that's $\omega(x) = 1 + x, x \in [-1, 1]$

Employing Eqs. (19)-(21), we obtain the following polynomials:

$$\begin{aligned}
 \chi_0(x) &= 1, \chi_1(x) = \frac{1}{2}(-1 + 3x), \chi_2(x) = \frac{1}{2}(-1 - 2x + 5x^2), \chi_3(x) = \frac{1}{8}(3 - 15x - 15x^2 + 35x^3), \\
 \chi_4(x) &= \frac{1}{8}(3 + 12x - 42x^2 - 28x^3 + 63x^4), \chi_5(x) = \frac{1}{16}(-5 + 35x + 70x^2 - 210x^3 - 105x^4 + 231x^5), \\
 \chi_6(x) &= \frac{1}{16}(-5 - 30x + 135x^2 + 180x^3 - 495x^4 - 198x^5 + 429x^6), \\
 \chi_7(x) &= \frac{1}{128}(35 - 315x - 945x^2 + 3465x^3 + 3465x^4 - 9009x^5 - 3003x^6 + 6435x^7), \\
 \chi_8(x) &= \frac{1}{128}(35 + 280x - 1540x^2 - 3080x^3 + 10010x^4 + 8008x^5 - 20020x^6 - 5720x^7 + 12155x^8), \\
 (23) \quad \chi_9(x) &= \frac{1}{256}(-63 + 693x + 2772x^2 - 12012x^3 - 18018x^4 + 54054x^5 + 36036x^6 - 87516x^7 - 21879x^8 \\
 &\quad + 46189x^9)
 \end{aligned}$$

For Case: $\alpha = 1$, $\beta = 1$, that's $\omega(x) = 1 - x^2$, $x \in [-1, 1]$

Employing Eqs. (19)-(21), we obtain the following polynomials:

$$\begin{aligned}
 \chi_0(x) &= 1, \chi_1(x) = x, \chi_2(x) = \frac{1}{4}(1 + 5x^2), \chi_3(x) = \frac{1}{4}(-3x + 7x^3), \\
 \chi_4(x) &= \frac{1}{8}(1 - 14x^2 + 21x^4), \chi_5(x) = \frac{1}{8}(5x - 30x^3 + 33x^5), \\
 \chi_6(x) &= \frac{1}{64}(-5 + 135x^2 - 495x^4 + 429x^6) \\
 (24) \quad \chi_7(x) &= \frac{1}{64}(-35x + 385x^3 - 1001x^5 + 715x^7), \\
 \chi_8(x) &= \frac{1}{128}(7 + -308x^2 + 2002x^4 - 4004x^6 + 2431x^8), \\
 \chi_9(x) &= \frac{1}{128}(63x - 1092x^3 + 4914x^5 - 7956x^7 + 4199x^9)
 \end{aligned}$$

We remark here that the polynomials of even order are even functions of x and those of odd order are odd functions of x for the polynomials corresponding to weight function $\omega(x) = 1 - x^2$ which is also a family of Gegenbauer polynomial $C_i^{(\gamma)}(x) = (1 - x^2)^{(\gamma - \frac{1}{2})}$ when $\gamma = \frac{3}{2}$, and this property is also exhibited by the κ -th kind Chebyshev polynomials because both belong to the same family called ultra-spherical polynomials.

We obtain the shifted form of the polynomial for the three cases using the transformation:

$$(25) \quad \begin{aligned} u &= \frac{1}{2}(b-a)x + \frac{1}{2}(a+b), -1 \leq x \leq 1 \\ u &= \frac{1}{2}(x+1) \text{ or } x = 2u-1, u \in [0, 1] \end{aligned}$$

It is convenient to have a special notation for this range, we now write:

$$(26) \quad \chi_i^*(x) = \chi_i(2x-1), 0 \leq x \leq 1.$$

All the properties of $\chi_i^*(x)$ can be deduced from $\chi_i(2x-1)$.

3.2. Description of the ν -recursive form. Lanczos noted that if a sequence of polynomials $Q_r(x), r = 0, 1, \dots, n$ such that

$$(27) \quad LQ_r(x) = x^r, \forall r \in \mathbb{N}$$

Can be found for any linear differential operator L with polynomial coefficients, since

$$(28) \quad \chi_n(x) = C_0^{(n)} + C_1^{(n)}x + \dots + C_n^{(n)}x^n = \sum_i^n C_i^{(n)}x^i$$

Then the solution of the ν -problem becomes:

$$(29) \quad y_n(x) = \nu_1 \sum_{r=0}^n C_r^{(n)} Q_r(x) + \nu_2 \sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(x) + \dots,$$

depending on the number of over-determination and the order of the differential equation (m) (as discussed in [5, 6]), the parameters are fixed using the given conditions.

An extension of this approach to a wider range of differential operator than the trivial one, has several advantages such as:

- (i) Canonical polynomials are independent of the interval in which the solution is sought, allowing for easy segmentation of the domain.
- (ii) They are permanent, in the sense that if an approximation of higher degree is required, the computation does not need to be repeated from scratch.
- (iii) They are also independent of the attached conditions

A recursive generation of Lanczos canonical polynomials was proposed by Ortiz in 1969 and it was generalized by Yisa [8] as:

(30)

$$Q_n(x) = \frac{1}{\sum_{k=0}^m k! \binom{n-s}{k} P_{k,k+s}} \left\{ x^{n-s} - \left[\sum_{k=1}^m \left(\sum_{j=k}^m j! \binom{n-s}{j} P_{j,j-k} \right) Q_{n-s-k}(x) \right. \right. \\ \left. \left. + \sum_{k=0}^{s-1} \left(\sum_{j=0}^m j! \binom{n-s}{j} P_{j,j+k} \right) Q_{n-s+k}(x) \right] \right\}$$

Let $y(x)$ be a known function which satisfied

$$(31) \quad Ly(x) = f(x)$$

where L is an n th order linear differential operator. We assume that $y(x)$, $a \leq x \leq b$ satisfies equation (1), the approximate solution to Eq. (1) using the recursive form was generalized by Issa [4, 6] as:

$$(32) \quad y_n(x) = \sum_{r=s}^{\sigma} f_r q_r(x) + \sum_{i=0}^{m+s-1} \nu_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} q_r(x)$$

(33)

$$y_n^\lambda(x) = \sum_{r=s}^{\sigma} f_r Q_r^\lambda(x) + \sum_{i=0}^{m+s-1} \nu_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} Q_r^\lambda(x) = \alpha_\lambda, \lambda = 0(1)(m-1)$$

$$(34) \quad \sum_{i=0}^{m+s-1} \nu_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0$$

where $q_n(x) = Q_n(x) - P_n$ and Eq. (34) is the coefficient of undetermined canonical polynomials,

$$(35) \quad P_n = \frac{-1}{\sum_{k=0}^m k! \binom{n-s}{k} P_{k,k+s}} \left\{ \sum_{k=1}^m \left(\sum_{j=k}^m j! \binom{n-s}{j} P_{j,j-k} \right) P_{n-s-k} \right. \\ \left. + \sum_{k=0}^{s-1} \left(\sum_{j=0}^m j! \binom{n-s}{j} P_{j,j+k} \right) P_{n-s+k} \right\}$$

4. NUMERICAL EXAMPLES

In this section, we implement the constructed polynomials generated from the family of weight function of Jacobi polynomial on some selected problems from the literature. We compute the maximum absolute error κ_N for each example and compare the results with the existing one. Where:

$$(36) \quad \kappa_N = \max_{0 \leq \ell \leq 100} |y(x_\ell) - y_N(x_\ell)|, \quad x_\ell = a + ih$$

Example 1. Consider the differential equation

$$\begin{aligned} y''(x) + y(x) &= 0, \\ y(0) &= 1, \end{aligned}$$

with the exact solution $y(x) = \cos x$.

Using Eq. (33), we obtain

$$(37) \quad \nu_1 \sum_{r=0}^n C_r^{(n)} Q_r(0) + \nu_2 \sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0) = 1$$

$$(38) \quad \nu_1 \sum_{r=0}^n C_r^{(n)} Q'_r(0) + \nu_2 \sum_{r=0}^{n-1} C_r^{(n-1)} Q'_r(0) = 0$$

Solving Eqs. (37) and (38) simultaneously for each cases of $\omega(x)$ at degree 5, 6, 7 and substitute the values of ν_1 and ν_2 in Eq. (32) to obtain the approximate solutions corresponding to each case of $\omega(x)$.

For $\omega(x) = 1 - x^2$, we obtain:

$$\begin{aligned} y_5(x) &= 1 - 0.4998555733x^2 - 2.1031402 \times 10^{-3}x^3 + 0.045935668x^4 - 3.94980064 \times 10^{-3}x^5 \\ y_6(x) &= 1 - 0.499986444x^2 - 8.7084842 \times 10^{-5}x^3 + 0.04192018x^4 - 3.284342161 \times 10^{-4}x^5 \\ &\quad - 1.207191243 \times 10^{-3}x^6 \\ y_7(x) &= 1 - 0.50000144x^2 + 1.3047658 \times 10^{-5}x^3 + 0.041611908x^4 + 1.21773071 \times 10^{-4}x^5 \\ &\quad - 1.537360513 \times 10^{-3}x^6 + 9.43342115 \times 10^{-5}x^7 \end{aligned}$$

For $\omega(x) = 1 + x$

$$\begin{aligned} y_5(x) &= 1 - 0.499275875x^2 - 2.9481893 \times 10^{-3}x^3 + 0.046916414x^4 - 4.341878935 \times 10^{-3}x^5 \\ y_6(x) &= 1 - 0.499974986x^2 - 1.40719211 \times 10^{-4}x^3 + 0.042019061x^4 - 4.247756816 \times 10^{-4}x^5 \\ &\quad - 1.175075583 \times 10^{-3}x^6 \\ y_7(x) &= 1 + -0.5000002477x^2 + 1.9972014 \times 10^{-5}x^3 + 0.041591155x^4 + 1.529058902 \times 10^{-4}x^5 \\ &\quad - 1.56019244 \times 10^{-3}x^6 + 1.008576284 \times 10^{-4}x^7 \end{aligned}$$

For $\omega(x) = 1 - x$, we obtain:

$$y_5(x) = 1 - 0.499849874x^2 - 1.2149728 \times 10^{-3}x^3 + 0.04490488x^4 - 3.53771666 \times 10^{-3}x^5$$

$$y_6(x) = 1 - 0.499996393x^2 - 4.0517269 \times 10^{-5}x^3 + 0.041819078x^4 - 2.447866242 \times 10^{-4}x^5 \\ - 1.23507537 \times 10^{-3}x^6$$

$$y_7(x) = 1 - 0.500000364x^2 + 5.8660376 \times 10^{-6}x^3 + 0.041633432x^4 + 8.948384722 \times 10^{-5}x^5 \\ - 1.51368051 \times 10^{-3}x^6 + 8.756848659 \times 10^{-5}x^7$$

Table 1 shows the maximum absolute errors at different values of n relative to their weight functions $\omega(x)$ figure 4a display the exact solution and its corresponding approximate solutions at various values of n for different cases while figure 4b is the comparison of the absolute errors.

TABLE 1. Maximum absolute errors for Example 1 at various values of n and $\omega(x)$

$\omega(x)$	$n = 5$	$n = 6$	$n = 7$
$\omega(x) = 1 - x^2$	2.75×10^{-4}	8.72×10^{-6}	4.34×10^{-8}
$\omega(x) = 1 + x$	4.82×10^{-5}	1.20×10^{-6}	2.14×10^{-8}
$\omega(x) = 1 - x$	1.76×10^{-6}	3.07×10^{-8}	1.90×10^{-9}

Example 2. Consider the differential equation [4]

$$y'(x) - x^2y(x) = 0, y(0) = 1,$$

with exact solution $y(x) = e^{\frac{x^3}{3}}$, $0 \leq x \leq 1$.

We obtain the following results at degree 5, 6 and 7 for different $\omega(x)$ as shown in Table 2 and figure 4.

TABLE 2. Maximum absolute errors for Example 2 at various values of n and $\omega(x)$

$\omega(x)$	$n = 5$	$n = 6$	$n = 7$
$\omega(x) = 1 - x^2$	2.63×10^{-4}	5.71×10^{-5}	6.78×10^{-6}
$\omega(x) = 1 + x$	5.03×10^{-4}	7.06×10^{-5}	7.65×10^{-6}
$\omega(x) = 1 - x$	3.64×10^{-4}	4.84×10^{-5}	8.30×10^{-6}

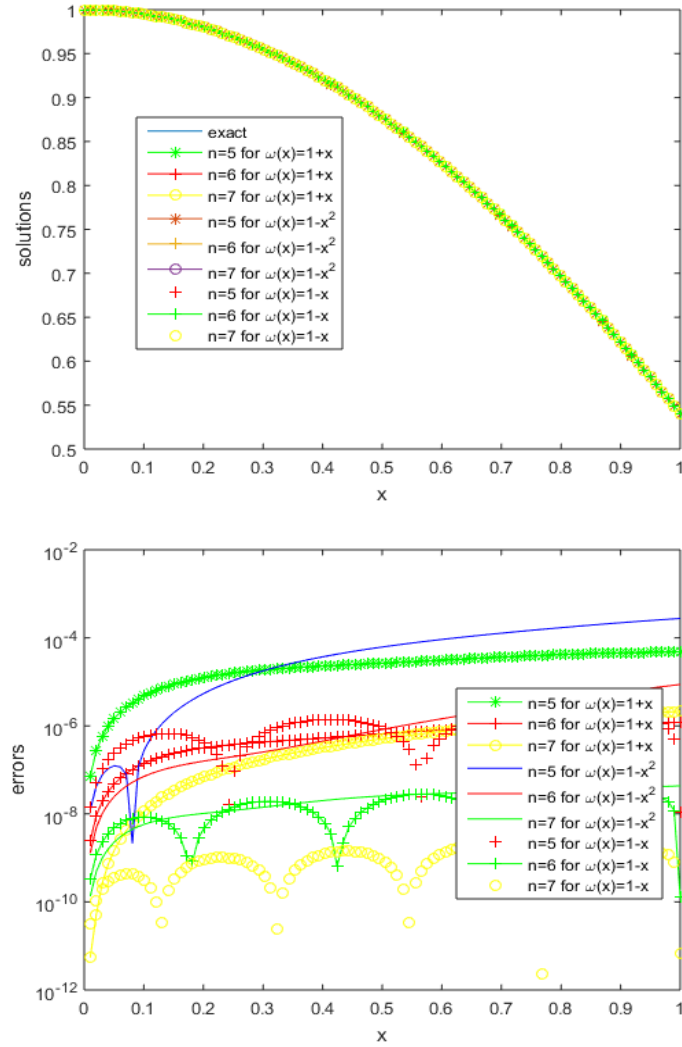


Figure 1: The exact and approximate solutions at various values of n and $\omega(x)$ for example 1

Example 3. Consider the differential equation [4]

$$2(1+x)y'(x) + y(x) = 0, \quad y(0) = 1, \quad 0 \leq x \leq 1$$

with exact solution $y(x) = (1+x)^{-\frac{1}{2}}$

The results corresponding the example 3 is shown the Table 3 and its corresponding figure is display in figure 4.

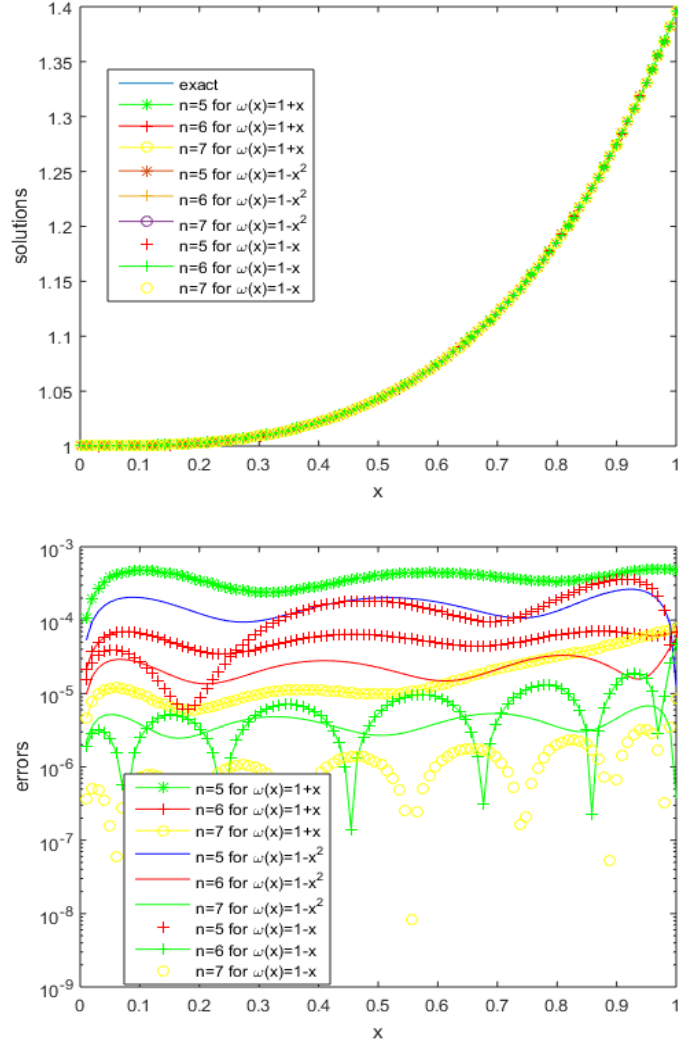


Figure 2: The exact and approximate solutions at various values of n and $\omega(x)$ for example 2

Discussion of Results. Table 1-3 depict maximum absolute errors obtained for the three selected problems from the literature. It was observed that the proposed technique give the same degree of error accuracy when compare with the recursive tau method in Table 4, therefore the significant of this research work is well established that the proposed method satisfies exactly the perturbed form of the differential equation under consideration. The accuracy improves as

TABLE 3. Maximum absolute errors for Example 3 at various values of n and $\omega(x)$

$\omega(x)$	$n = 5$	$n = 6$	$n = 7$
$\omega(x) = 1 - x^2$	4.01×10^{-5}	5.95×10^{-6}	1.20×10^{-7}
$\omega(x) = 1 + x$	6.70×10^{-5}	7.17×10^{-6}	9.02×10^{-7}
$\omega(x) = 1 - x$	2.28×10^{-5}	8.12×10^{-6}	8.06×10^{-7}

TABLE 4. Results from [4]: Maximum absolute errors for Examples 2 and 3

Examples	$n = 5$	$n = 6$	$n = 7$
Example 2	2.07×10^{-4}	2.96×10^{-5}	4.50×10^{-6}
Example 3	3.80×10^{-5}	4.24×10^{-6}	6.64×10^{-7}

values of n increases. Figures 4-4 are the exact solutions and its corresponding approximate solution at various values of n while 4b-4b are the error graphs.

Conclusions: In this paper, we proposed a technique analogue to the recursive tau method for solving differential equation. The method was implemented on some selected problems from the literature, the results were then compared with the existing results. The results obtain show the effectiveness, accuracy of proposed technique and the significant of the proposed technique as it give the same degree of accuracy with results obtained using recursive tau method.

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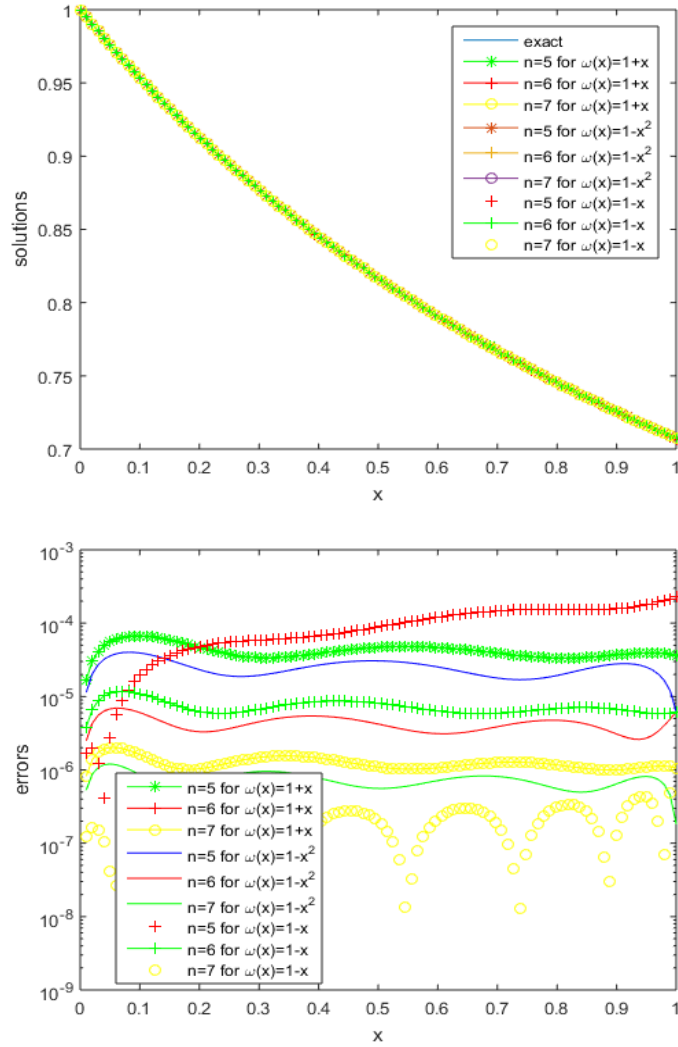


Figure 3: The exact and approximate solutions at various values of n and $\omega(x)$ for example 3

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