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Extension of generalized recursive Tau method to non-linear ordinary differential equations

K. Issa[a,b,*], R.B. Adeniyi[a]

[a] *Department of Mathematics, University of Ilorin, Nigeria*
[b] *Department of Statistics and Mathematical Sciences, Kwara State University, Malete, Ilorin, Nigeria*

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Abstract

In a recent paper, we reported a generalized approximation technique for the recursive formulation of the Tau method. This paper is concerned with an extension of that discourse to non-linear ordinary differential equations. The numerical results show that the method is effective and accurate.
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1. Introduction

In 1981, Ortiz and Samara [1] proposed an operational technique for the numerical solution of non-linear ordinary differential equations with some supplementary conditions based on the Tau method [2]. Recently, considerable work has been done both in the development of the technique, its theoretical analysis and numerical applications. The technique has been described in a series of papers [3–6,1,12–15], for the case of linear ordinary differential eigenvalue problems. Yisa and Adeniyi [7] reported the construction of generalized canonical polynomials while Issa and Adeniyi [8,16] reported generalized approximation for the recursive formulation of the Tau method for the solution of ordinary differential equations, their earlier works are further extended to non-linear ordinary differential equations.

2. Recursive formulation of Tau approximant

In this section, we review the Tau approximant for the recursive form (see [8]) using the generalized Canonical polynomials $Q_n(x)$ (see [7]) to solve the mth order ordinary differential equation of the form:

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* Corresponding author at: Department of Mathematics, University of Ilorin, Nigeria.
E-mail addresses: issakazeem@yahoo.com (K. Issa), raphade@unilorin.edu.ng (R.B. Adeniyi).

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$$Ly(x) \equiv \sum_{r=0}^{m} \left(\sum_{k=0}^{N_r} P_{r,k} x^k \right) y^{(r)}(x) = \sum_{r=0}^{\sigma} f_r x^r, \quad a \le x \le b \tag{2.1a}$$

$$L^* y(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \quad k = 1(1)m \tag{2.1b}$$

by seeking an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad r < +\infty$$

of $y(x)$ which is the exact solution of the corresponding perturbed system

$$Ly_n(x) = \sum_{r=0}^{\sigma} f_r x^r + H_n(x) \tag{2.2a}$$

$$L^* y_n(x_{rk}) = \alpha_k, \quad k = 1(1)m \tag{2.2b}$$

where α_k, f_k, $P_{r,k}$, N_r; $r = 0(1)m$, $k = 0(1)N_r$ are real integers, $y^{(r)}$ denote the derivatives of order r of $y(x)$, the perturbation term $H_n(x)$ in (2.2a) is defined by:

$$H_n(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} T_{n-m+i+1}(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} x^r \tag{2.2c}$$

and $C_r^{(n)}$ is the coefficient of x^r in the nth degree Chebyshev polynomial $T_n(x)$; that is,

$$T_n(x) = \cos\left(n \cos^{-1}\left\{ \frac{2x - a - b}{b - a} \right\} \right) \equiv \sum_{r=0}^{n} C_r^{(n)} x^r.$$

The τ's are fixed parameters to be determined and s, the number of overdetermination of (2.1a), is defined by:

$$s = max\,\{N_r - r > 0 \mid 0 \le r \le m\}.$$

For different orders m and s (that is $m = 1, 2, \ldots$ and $s = 1, 2, \ldots$) we have

$$y_n(x) = \sum_{r=s}^{\sigma} f_r q_r(x) + \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} q_r(x). \tag{2.3a}$$

Assume $Q_r(x) = P_r = 1, r = 0(1)(s-1)$ and

$$\sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0 \tag{2.3b}$$

where Eq. (2.3b) is the coefficient of undetermined Canonical polynomials, $q_n(x) = Q_n(x) - P_n$,

$$P_n = \frac{-1}{\sum_{k=0}^{m} k! \binom{n-s}{k} P_{k,k+s}} \left\{ \sum_{k=1}^{m} \left(\sum_{j=k}^{m} j! \binom{n-s}{j} P_{j,j-k} \right) P_{n-s-k} + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j! \binom{n-s}{j} P_{j,j+k} \right) P_{n-s+k} \right\} \tag{2.3c}$$

we assume $P_r = 1,\ r = 0,\ 1\ \ldots s-1$, when equating the coefficient of $Q_r(x)$ to zero, otherwise $P_r = 0$, $P_r = 1,\ r = 0,\ 1\ \ldots s-1$, see [8] and

$$Q_n(x) = \frac{1}{\sum_{k=0}^{m} k! \binom{n-s}{k} P_{k,k+s}} \left\{ x^{n-s} - \left[\sum_{k=1}^{m} \left(\sum_{j=k}^{m} j! \binom{n-s}{j} P_{j,j-k} \right) Q_{n-s-k}(x) + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j! \binom{n-s}{j} P_{j,j+k} \right) Q_{n-s+k}(x) \right] \right\} \tag{2.3d}$$

is the generalized Canonical polynomial, see [7].

3. Methodology

In this section, we shall consider an extension of the generalized recursive formulation of the Tau method to non-linear problem. For this purpose, we shall employ the Newton–Kantorovich linearization process to non-linear differential equation of the form

$$G(x, y(x), y'(x), \ldots, y^{(m)}(x)) = \sum_{r=0}^{\sigma} f_r x^r \tag{3.1}$$

and the process of Newton–Kantorovich linearization, derived from the Taylor series expansion in several variables of G, is given by:

$$G + \triangle y \frac{\partial G}{\partial y} + \triangle y' \frac{\partial G}{\partial y'} + \triangle y'' \frac{\partial G}{\partial y''} + \cdots + \triangle y^{(m)} \frac{\partial G}{\partial y^{(m)}} = \sum_{r=0}^{\sigma} f_r x^r \tag{3.2}$$

where $\triangle y_k^i = y_{k+1}^i - y_k^i$, $i = 0, 1, \ldots, m$.

We seek kth iterative approximant of the form:

$$y_{n,k}(x) = \sum_{r=0}^{n} a_{r,k} x^r. \tag{3.3}$$

The form (2.2) corresponding to Eq. (3.2) is

$$\sum_{j=0}^{m} (y_{n,k+1}^j(x) - y_{n,k}^j(x)) \frac{\partial G_{n,k}}{\partial y_{n,k}^j} = \sum_{r=0}^{\sigma} f_r x^r - G_{n,k} + H_{n,k}(x) \tag{3.4}$$

where

$$H_{n,k}(x) = \sum_{i=0}^{m+s-1} \tau_{i+1,k} T_{n-m+i+1}(x) \tag{3.5}$$

and

$$G_{n,k} \equiv G(x, y_{n,k}(x), y'_{n,k}(x), \ldots, y_{n,k}^{(m)}(x)), \quad k = 0, 1, \ldots. \tag{3.6}$$

The number of overdetermination s, for the non-linear problems, unlike in the case of linear problems, depends on n and can be very large depending on the degree of non-linearity of the problem under consideration. The iterative process is repeated until

$$| \xi_{n,k} - \xi_{n,k+1} | \ \leq \text{Tolerance Value}$$

where

$$\xi_{n,k} = \max\{|y_k(x) - y_{n,k}(x)| : a \leq x \leq b\}. \tag{3.7}$$

The iterative scheme needs a suitable choice of an initial approximation $y_{n,0}(x)$ from linearization process, for a rapid convergence. In most problems, the initial approximation $y_{n,0}(x)$ is taken to be the simplest polynomial satisfying the associated condition in (2.1b). In some cases, good choice of initial approximation is obtained from the given differential equation itself (see Example 4.1).

4. Numerical examples

We shall consider here four problems of interest for the illustration of the method of the preceding section. It is to be noted that the results presented in the Tables below were obtained using mathematica 7.0 package.

We illustrate the accuracy of the presented method by computing $\xi_{n,k}$ (maximum error) for various numerical examples.

Example 4.1 (*Variable Coefficients First Order Homogeneous Problem, see [9]*)**.** Consider the first order non-linear problem:

$$y'(x) + (2x-1)y^2(x) = 0 \tag{4.1}$$

$$y(0) = 1, \quad 0 \le x \le 1 \tag{4.2}$$

with exact solution $y(x) = (x^2 - x + 1)^{-1}$.

For this problem, we have

$$G_k \equiv G(x, y_k(x), y'_k(x)) = y'_k(x) + (2x-1)y_k^2(x)$$

so that

$$G_k + \triangle y_k \frac{\partial G_k}{\partial y_k} + \triangle y'_k \frac{\partial G_k}{\partial y'_k} = 0 \tag{4.3}$$

where

$$\triangle y_k = y_{k+1} - y_k, \qquad \triangle y'_k = y'_{k+1} - y'_k, \qquad \frac{\partial G_k}{\partial y_k} = 2y_k(2x-1), \qquad \frac{\partial G_k}{\partial y'_k} = 1$$

substitute in (4.3), leads to the linearized problem

$$y'_{k+1} + (4x-2)y_k y_{k+1} = (2x-1)y_k^2 \tag{4.4}$$

from (3.4), we have

$$y'_{n,k+1} + (4x-2)y_{n,k}y_{n,k+1} = (2x-1)y_{n,k}^2 + H_{n,k}(x)$$

where

$$H_{n,k}(x) = \sum_{i=0}^{s} \tau_{i+1,k} T_{n+i}(x).$$

Thus, the sequence of linearized Tau problem to be solved is:

$$y'_{n,k+1} + (4x-2)y_{n,k}y_{n,k+1} = (2x-1)y_{n,k}^2 + \sum_{i=0}^{s} \tau_{i+1,k} T_{n+i}(x) \tag{4.5a}$$

$$y_{n,k+1}(0) = 1, \quad k = 0, 1, 2, \ldots. \tag{4.5b}$$

For the choice of initial approximation, we have from (4.1), $y'(0) = 1$ (since $y(0) = 0$). Hence, if we assume an initial approximation of the form $y = a + bx$, then we get $y(x) = x + 1$ by using $y(0) = 0$ and $y'(0) = 1$ to determine a and b. So, we choose $y_{n,0}(x) = x + 1$ and then compute the approximant solution.

First Iteration (k=0): we have from (4.4)

$$y'_1(x) + (4x^2 + 2x - 2)y_1(x) = 2x^3 + 3x^2 - 1 \tag{4.5}$$

$$y'_1(0) = 1.$$

Table 4.1
Maximum error for Example 4.1.

Iteration (k)	Degree 5 ($\xi_{5,k}$)	Degree 6 ($\xi_{6,k}$)	Degree 7 ($\xi_{7,k}$)
$k=0$	9.37×10^{-2}	9.39×10^{-2}	9.41×10^{-2}
$k=1$	2.23×10^{-3}	4.64×10^{-4}	4.44×10^{-4}
$k=2$	2.27×10^{-3}	1.67×10^{-4}	1.70×10^{-4}

Comparing (4.5) with (2.1a), we have $m=1\, s=2$, and so the perturbed problem (4.5) becomes

$$y'_{n,1}(x)+(4x^2+2x-2)y_{n,1}(x)=2x^3+3x^2-1+\tau_1T_n(x)+\tau_2T_{n+1}(x)+\tau_3T_{n+2}(x).$$

Applying (2.3), we obtained the solution for degrees 5, 6 and 7 as:

$$y_{5,1}(x)=1+\frac{13734241x}{14036168}+\frac{2477864x^2}{5263563}-\frac{15478136x^3}{5263563}+\frac{2400320x^4}{1754521}+\frac{1142176x^5}{5263563}$$

$$\begin{aligned}y_{6,1}(x) &= 1+\frac{758745347x}{754661830}-\frac{1599024368x^2}{7923949215}+\frac{923988736x^3}{1584789843}-\frac{15803064192x^4}{2641316405}\\ &\quad+\frac{54946095104x^5}{7923949215}-\frac{5926750208x^6}{2641316405}\end{aligned}$$

$$\begin{aligned}y_{7,1}(x) &= 1+\frac{109326648889x}{109205132560}-\frac{643141379x^2}{13650641570}-\frac{1645570879x^3}{2730128314}-\frac{15100542224x^4}{6825320785}\\ &\quad+\frac{7700535256x^5}{6825320785}+\frac{14043641024x^6}{6825320785}-\frac{1679799488x^7}{1365064157}\end{aligned}$$

respectively. **Second Iteration (k=1)**: We similarly obtained

$$y_{5,2}(x)=1+0.982633448x+0.398369665x^2-2.746545092x^3+1.345127453x^4+0.020877660x^5$$

$$\begin{aligned}y_{6,2}(x) &= 1+1.001540930x-0.073718784x^2-0.27552491x^3-3.801892850x^4\\ &\quad+4.714527741x^5-1.564467919x^6\end{aligned}$$

$$\begin{aligned}y_{7,2}(x) &= 1+1.001763174x-0.081717596x^2-0.214195149x^3-3.996928214x^4\\ &\quad+5.014854129x^5-1.786985423x^6+0.0636669636x^7\end{aligned}$$

and for **Third Iteration (k=2)** we have

$$y_{5,3}(x)=1+0.982318x+0.405095x^2-2.77481x^3+1.3874x^4+5.28099\times10^{-6}x^5$$

$$\begin{aligned}y_{6,3}(x) &= 1+1.001699849x-0.078836193x^2-0.241731719x^3-3.889122757x^4+4.811985927x^5\\ &\quad-1.603995100x^6\end{aligned}$$

$$\begin{aligned}y_{7,3}(x) &= 1+1.001699864x-0.0788367194x^2-0.241727690x^3-3.889135527x^4+4.812005511x^5\\ &\quad-1.60401x^6+4.111601230\times10^{-6}x^7\end{aligned}$$

respectively.

Table 4.1 is the maximum error for each iteration, and this is in good agreement with the result obtained in the literature [9].

Example 4.2 (*A Constant Coefficients Second Order Homogeneous Problem, see [10]*)**.**

$$y''(x)-y(x)y'(x)=0$$

$$y(0)=0,\qquad y(1)=\tanh\left(\frac{-1}{2}\right)$$

with analytic solution $\tanh(\frac{-x}{2})$, $0\le x\le1$ after linearized, we have

$$y''_{k+1}-y_ky'_{k+1}-y'_ky_{k+1}=-y_ky'_k$$

Table 4.2
Maximum error for Example 4.2.

Iteration (k)	$\xi_{5,k}$	$\xi_{6,k}$	$\xi_{7,k}$
$k=0$	1.14×10^{-5}	9.22×10^{-6}	9.42×10^{-6}
$k=1$	5.91×10^{-6}	2.64×10^{-7}	3.95×10^{-8}
$k=2$	5.91×10^{-6}	2.64×10^{-7}	3.99×10^{-8}

Table 4.3
Maximum error for Example 4.3.

Iteration (k)	$\xi_{5,k}$	$\xi_{6,k}$	$\xi_{7,k}$
$k=0$	5.18×10^{-3}	5.17×10^{-3}	5.16×10^{-3}
$k=1$	1.31×10^{-4}	2.86×10^{-5}	3.77×10^{-6}
$k=2$	1.31×10^{-4}	2.39×10^{-5}	3.78×10^{-6}

$$y_{k+1}(0)=0, \qquad y_{k+1}=\tanh\left(\frac{-1}{2}\right), \quad 0\le x\le 1,$$

see Table 4.2 for the maximum error.

Example 4.3 (*A Constant Coefficients First Order Non-homogeneous Problem, see [9]*)**.**

$$y'(x)-(y(x))^2=1$$
$$y(0)=1$$

with analytic solution $y(x)=\tan x$, $0\le x\le\frac{\pi}{4}$.

This leads to the linearized problem

$$y'_{k+1}-2y_k y_{k+1}=1-y_k^2$$
$$y_{k+1}(0)=0$$

see Table 4.3 for the maximum error.

Example 4.4. Consider the linear initial value problem $y''(x)+2y'(x)+y(x)=0,\ y(0)=1,\ y'(0)=0$.

The closed form of the solution is $y(x)=e^{-x}(1+x)$ and the approximate solution using the Adomian decomposition method (see [11]) is:

$$y(x)=1-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{8}x^4+\frac{1}{30}x^5+O(6).$$

The presented method gives the following results:

$$y_5(x)=1-\frac{1510240}{3023161}x^2+\frac{2978560}{9069483}x^3-\frac{3032960}{27208449}x^4+\frac{499712}{27208449}x^5$$
$$y_6(x)=1-\frac{19830276760}{39662689081}x^2+\frac{13202664320}{39662689081}x^3-\frac{4880385280}{39662689081}x^4+\frac{1181708288}{39662689081}x^5-\frac{154245120}{39662689081}x^6$$
$$y_7(x)=1-\frac{40269705965496}{80539624942061}x^2+\frac{26843943252224}{80539624942061}x^3-\frac{10051654400256}{80539624942061}x^4+\frac{2642224435200}{80539624942061}x^5$$
$$-\frac{501023258624}{80539624942061}x^6+\frac{54334488576}{80539624942061}x^7$$

see Table 4.4 for the maximum error.

Table 4.2 is the maximum error for each iteration, and this gives better result compared to the result obtained by the cubic spline using collocation method (see [10]).

Table 4.3 is the maximum error for each iteration, and this is in good agreement with the result obtained in the literature [9].

Table 4.4
Maximum error for Example 4.4.

Degree	Presented method	Adomian decomposition method
5	2.65×10^{-5}	5.91×10^{-3}
6	8.19×10^{-7}	
7	3.07×10^{-8}	

Table 4.4 is the maximum error, and this gives better result compared to the result obtained by the Adomian decomposition method (see [11]).

5. Conclusion

The generalized form of the recursive formulation of the Tau method for initial value problems and boundary value problems has been applied to non-linear problems and to linear differential equation, we obtained the maximum error for degrees 5, 6 and 7. Our results were compared with some existing results and we found that they are in good agreement.

Finally, because of the accuracy and simplicity of the method presented in this study, we recommend its application in finding approximate solution to non-linear differential equations.

References

[1] Ortiz EL, Samara H. An operational approach to the Tau method for the numerical solution of non-linear differential equations. Computing 1981;27:15–25.
[2] Ortiz EL. The Tau method. SIAM J Numer Anal 1969;6:480–92.
[3] Liu KM, Ortiz EL. Numerical solution of Ordinary and Partial function-differential eigenvalue problems with the Tau method. Computing, (wien) 1989;41:205–17.
[4] Liu KM, Ortiz EL. Tau method approximation of differential eigenvalue problems where the spectral parameter enters non-linearly. J Comput Phys 1987;72:299–310.
[5] Liu KM, Ortiz EL. Approximation of eigenvalues defined by ordinary differential equations with the Tau method. In: Ka gestrm B, Ruhe A, editors. Matrix pencils. Berlin: Springer-Verlag; 1983. p. 90–102.
[6] Liu KM, Ortiz EL. Eigenvalue problems for singularly perturbed differential equations. In: Miller JJH, editor. Proceedings of the BAIL II conference. Dublen: Boole Press; 1982. p. 324–9.
[7] Yisa BM, Adeniyi RB. Generalization of canonical polynomials for overdetermined *m*th order ordinary differential equations(ODEs). IJERT 2012;1(6):1–15.
[8] Issa K, Adeniyi RB. A generalized scheme for the numerical solution of initial value problems in ordinary differential equations by the recursive formulation of Tau method. Int J Pure Appl Math 2013;88(1):1–13.
[9] Adeniyi RB, Onumanyi P. Error estimation in the numerical solution of ODE with the Tau method. Comput Math Appl 1991;21(9):19–27.
[10] Taiwo OA. Comparison of collocation methods for the solution of second order non-linear boundary value problems. Int J Comput Math 2005;82(11):1389–401.
[11] Onur Kymaz, Seref Mirasyedioglu. A new symbolic computational approach to singular initial value problems in the second order ordinary. Appl Math Comput 2005;171:1218–25. differential equations.
[12] Adeniyi RB, Edungbola EO. On the tau method for certain overdetermined first order differential equations. J Niger Assoc Math Phys Soc 2008;12:399–408.
[13] Adeniyi RB, Edungbola EO. On the recursive formulation of the tau method for class of overdetermined first order equations. Abacus J Math Assoc Nig 2007;34(2B):249–61.
[14] Fox L, Parker IB. Chebyshev Polynomials in numerical analysis. University Press Oxford; 1968.
[15] Lanczos C. Trigonometric interpolation of empirical and analytic functions. J Math Phys 1938;17:123–99.
[16] Ortiz EL, Samara H. Numerical solution of differential eigenvalue problems with an operational approach to the Tau method. Computing 1983;31:95–103.