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ON THE EXISTENCE OF FIXED POINTS VIA CONDENSED REICH-RUS-ĆIRIĆ-TYPE CONTRACTIONS

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Abstract. Many research papers have recently studied the interpolation technique of contractive-type maps to examine the existence of nonlinear operators that do not require unique fixed points. In this article, we introduce a notion of condensed Reich-Rus-Ćirić-type (CRRC) contraction to examine the existence of operators that require both unique and non-unique fixed points. By condensing the CRRC map, we establish and prove some fixed point theorems in standard metric spaces. An illustrative example is considered to show that the new CRRC-type map exhibits contracting behaviors when others do not. The study concludes that the new CRRC-type map improves and includes many known Reich-Rus-Ćirić-type maps in the literature.

Keywords. Fixed point, Reich-Rus-Ćirić-type contraction, condensed Reich-Rus-Ćirić-type map, metric spaces
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1. Introduction and preliminaries

Since the Banach principle [2] became a fundamental statement that serves as a benchmark in the classical fixed point theory, many mathematical authors have explored the applicability of the principle in different types of spaces and have deployed its modified versions to solve specific problems in applied science and engineering.

Theorem 1.1. [2] *Let (Ξ, ρ) be a complete metric space and $\Phi : \Xi \to \Xi$ be a Banach contraction mapping*

$$\rho(\Phi\upsilon, \Phi\omega) \leq a\rho(\upsilon, \omega) \tag{1.1}$$

for $a \in [0, 1)$ and $\upsilon, \omega \in \Xi$. Then, Φ has a unique fixed point.

One of the earliest modified versions of inequality (1.1) is the Kannan contraction [6]. Kannan introduced a class of mappings that are not necessarily continuous to prove the following theorem.

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Theorem 1.2. [6] *Let (Ξ, ρ) be a complete metric space and $\Phi : \Xi \to \Xi$ be a Kannan mapping*

$$\rho(\Phi\upsilon, \Phi\omega) \leq a\left[\rho(\upsilon, \Phi\upsilon) + \rho(\omega, \Phi\omega)\right] \tag{1.2}$$

for $a \in [0, \frac{1}{2})$ and $\upsilon, \omega \in \Xi$. Then, Φ has a unique fixed point.

Another independent fixed point theorem, which harmonized both Banach and Kannan results, was stated as follows:

Theorem 1.3. [10] *Let (Ξ, ρ) be a complete metric space. If a self-map $\Phi : \Xi \to \Xi$ satisfies*

$$\rho(\Phi\upsilon, \Phi\omega) \leq b\left[\rho(\upsilon, \omega) + \rho(\upsilon, \Phi\upsilon) + \rho(\omega, \Phi\omega)\right] \tag{1.3}$$

for $b \in [0, \frac{1}{3})$ and $\upsilon, \omega \in \Xi$. Then, Φ has a unique fixed point.

We note that condition (1.3) is a Reich-Rus-Ćirić contraction since Theorem 1.3 was proved independently by the trio. In 2018, Karapinar [7] defined the notion of interpolative Kannan-type contraction by modifying the Kannan contraction (1.2) as follows:

Definition 1.4. [7] Let (Ξ, ρ) be a metric space. A map $\Phi : \Xi \to \Xi$ is called an interpolative Kannan-type contraction if there exist $\alpha \in (0, 1)$ and $\mu \in [0, 1)$ such that

$$\rho(\Phi\upsilon, \Phi\omega) \leq \mu\rho(\upsilon, \Phi\upsilon)^{\alpha}\rho(\omega, \Phi\omega)^{1-\alpha} \tag{1.4}$$

for all $\upsilon, \omega \in \Xi$.

Any mapping Φ satisfying (1.4) has a unique fixed point in Ξ. However, authors in [8] noted that only the map Φ satisfying (1.4) such that $\upsilon, \omega \in \Xi \setminus Fix(\Phi)$ has a unique fixed point. Since then, numerous researchers have deployed several interpolative-type techniques to study the existence properties of non-unique fixed point problems, see [1, 4, 9, 5, 13]. Also in [8], the authors introduced the notion of interpolative Reich-Rus-Ćirić type (IRRC) condition in partial and standard metric spaces. Their result, in a standard metric space, is presented as follows:

Theorem 1.5. [8] *Let (Ξ, ρ) be a complete metric space. If for all $\upsilon, \omega \in \Xi \setminus Fix(\Phi)$, a self-map $\Phi : \Xi \to \Xi$ satisfies*

$$\rho(\Phi\upsilon, \Phi\omega) \leq d\rho(\upsilon, \omega)^{\alpha}\rho(\upsilon, \Phi\upsilon)^{\beta}\rho(\omega, \Phi\omega)^{1-\alpha-\beta} \tag{1.5}$$

where $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ and $d \in [0, 1)$. Then, Φ possesses a fixed point in Ξ.

See also [1, 4, 9, 11] for other versions of Theorem 1.5. Recently, in [12, 13], the authors introduced the notion of Kannan-type contraction by condensation and developed a technique for determining the suitable exponent to approximate fixed point problems.

Despite the improvement in Ref. [1, 4, 9, 11], we observe through experimentation that there are several mappings (continuous or not) that do not satisfy the IRRC (1.5) for some parametric exponents α, β. Therefore, this study aims to introduce a more robust RRC-type condition that ensures both unique and non-unique fixed points, and to prove existence theorems for nonlinear operators in standard spaces. We shall adopt the following lemmas in the sequel.

Lemma 1.6. [3] *Let σ, ς, τ be three non-negative real numbers, then following holds*

$$(\sigma+\varsigma+\tau)^3 \geq \sigma^3+\varsigma^3+\tau^3+24\sigma\varsigma\tau$$

Lemma 1.7. *A sequence υ_n in (Ξ,ρ) will be called a Cauchy sequence if for each $m\in\mathbb{N}$, we have*

$$\lim_{n\to\infty}\rho(\upsilon_{n+m},\upsilon_n)=0,\ \forall\ n\in\mathbb{N}.$$

2. Main Results

This section introduces and proves the concept of Condensed Reich-Rus-Ćirić-type (CRRC) contraction in complete metric spaces.

Definition 2.1. Let (Ξ,ρ) be a metric space. A self-mapping Φ on Ξ is called a CRRC-type contraction if there exist $\lambda\in[0,\frac{1}{3})$, $\mu\in[0,1)$, and positive real numbers α,β with $\alpha+\beta<1$ such that

$$\rho(\Phi\upsilon,\Phi\omega)\begin{cases}\geq\mu\rho(\upsilon,\omega)^{\alpha}\rho(\upsilon,\Phi\upsilon)^{\beta}\rho(\omega,\Phi\omega)^{1-\alpha-\beta} & \text{(CRRC1)}\\ \\ \leq\lambda\left[\rho(\upsilon,\omega)^{3\alpha}+\rho(\upsilon,\Phi\upsilon)^{3\beta}+\rho(\omega,\Phi\omega)^{3(1-\alpha-\beta)}\right] & \text{(CRRC2)}\end{cases}\tag{2.1}$$

for all $\upsilon,\omega\in\Xi\setminus\mathrm{Fix}(\Phi)$.

The following set of assumptions (to be justified in the sequel) is made from Definition 2.1.

i. The terms $\rho(\Phi\upsilon,\Phi\omega)$, CRRC1, and CRRC2 do not overlap, and gaps in between demonstrate the flexibility allowed by the parameters λ, μ, α, β.
ii. The distance map $\rho(\Phi\upsilon,\Phi\omega)$ lies between CRRC1 and CRRC2.
iii. If CRRC1 and CRRC2 define a region where $\rho(\Phi\upsilon,\Phi\omega)$ must lie, the map Φ possesses a fixed point.

The following theorem is one of the results of this study.

Theorem 2.2. *Let (Ξ,ρ) be a complete metric space and $\Phi:\Xi\to\Xi$ be a CRRC-type contraction for $\lambda\in(0,\frac{1}{3})$, $\mu\in(0,1)$, and $\alpha,\beta\in(0,1)$ such that $\alpha+\beta<1$. Then, Φ has a unique fixed point in Ξ.*

Proof. For a generic point $\upsilon_0\in\Xi\setminus\mathrm{Fix}(\Phi)$ and sequence $\{\upsilon_n\}\subset\Xi\setminus\mathrm{Fix}(\Phi)$, define $\upsilon_{n+1}=\Phi\upsilon_n$ for each non-negative integer n. If there exists n_0 such that $\upsilon_{n_0}=\upsilon_{n_0+1}$, then υ_{n_0} is a fixed point of Φ. This completes the proof. Assume that $\upsilon_n\neq\upsilon_{n+1}$ for each $n\geq 0$. By using Lemma 1.6 and letting $\upsilon=\upsilon_{n-1}$, $\omega=\upsilon_n$ in (CRRC2) of (2.1), we have

$$\begin{aligned}\rho(\upsilon_n,\upsilon_{n+1})&=\rho(\Phi\upsilon_{n-1},\Phi\upsilon_n)\\ &\leq\lambda\left[\rho(\upsilon_{n-1},\upsilon_n)^{3\alpha}+\rho(\upsilon_{n-1},\Phi\upsilon_{n-1})^{3\beta}+\rho(\upsilon_n,\Phi\upsilon_n)^{3(1-\alpha-\beta)}\right]\\ &=\lambda\left[\rho(\upsilon_{n-1},\upsilon_n)^{3\alpha}+\rho(\upsilon_{n-1},\upsilon_n)^{3\beta}+\rho(\upsilon_n,\upsilon_n+1)^{3(1-\alpha-\beta)}\right]\\ &\leq\lambda\Big\{\left[\rho(\upsilon_{\eta-1},\upsilon_n)^{\alpha}+\rho(\upsilon_{n-1},\upsilon_\eta)^{\beta}+\rho(\upsilon_n,\upsilon_{n+1})^{(1-\alpha-\beta)}\right]^3\\ &\qquad -24\rho(\upsilon_{n-1},\upsilon_n)^{\alpha+\beta}\rho(\upsilon_n,\upsilon_{n+1})^{(1-\alpha-\beta)}\Big\}\end{aligned}\tag{2.2}$$

More so from (CRRC1) of (2.1), we obtain

$$\rho(\upsilon_n, \upsilon_{n+1}) \geq \mu\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha+\beta}\rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)} \tag{2.3}$$

By combining (2.2) and (2.3) using $\mu \equiv 3\lambda \in (0, 1)$, we have

$$\begin{aligned} 27\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha+\beta}&\rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)} \\ &\leq \left(\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha} + \rho(\upsilon_{n-1}, \upsilon_n)^{\beta} + \rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)}\right)^3 \end{aligned}$$

This further becomes

$$\begin{aligned} \rho(\upsilon_{n-1}, \upsilon_n)^{\frac{\alpha+\beta}{3}}&\rho(\upsilon_n, \upsilon_{n+1})^{\frac{(1-\alpha-\beta)}{3}} \\ &\leq \frac{\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha} + \rho(\upsilon_{n-1}, \upsilon_n)^{\beta} + \rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)}}{3} \end{aligned} \tag{2.4}$$

We deduce three inequalities from (2.4) as follows:
I. For all $\alpha, \beta \in (0, 1)$, if

$$\begin{aligned} &\frac{\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha} + \rho(\upsilon_{n-1}, \upsilon_n)^{\beta} + \rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)}}{3} \\ &\qquad \leq \max_{\alpha,\beta}\left\{\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha}, \rho(\upsilon_{n-1}, \upsilon_n)^{\beta}, \rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)}\right\} = \rho(\upsilon_{n-1}, \upsilon_n)^{\alpha}, \end{aligned}$$

we have, by transitivity, that

$$\rho(\upsilon_{n-1}, \upsilon_n)^{\frac{\alpha+\beta}{3}}\rho(\upsilon_n, \upsilon_{n+1})^{\frac{(1-\alpha-\beta)}{3}} \leq \rho(\upsilon_{n-1}, \upsilon_n)^{\alpha}$$

which further gives

$$\rho(\upsilon_n, \upsilon_{n+1})^{1-\alpha-\beta} \leq \rho(\upsilon_{n-1}, \upsilon_n)^{2\alpha-\beta}. \tag{2.5}$$

II. If $\rho(\upsilon_{n-1}, \upsilon_n)^{\beta} = \max\left\{\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha}, \rho(\upsilon_{n-1}, \upsilon_n)^{\beta}, \rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)}\right\}$, we have from inequality (2.4) that

$$\rho(\upsilon_n, \upsilon_{n+1})^{1-\alpha-\beta} \leq \rho(\upsilon_{n-1}, \upsilon_n)^{2\beta-\alpha}. \tag{2.6}$$

III. Also, if $\rho(\upsilon_{n-1}, \upsilon_n)^{1-\alpha-\beta} = \max\left\{\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha}, \rho(\upsilon_{n-1}, \upsilon_n)^{\beta}, \rho(\upsilon_n, \upsilon_{n+1})^{(1-\alpha-\beta)}\right\}$, then

$$\rho(\upsilon_{n-1}, \upsilon_n)^{\alpha+\beta} \leq \rho(\upsilon_n, \upsilon_{n+1})^{2(1-\alpha-\beta)}. \tag{2.7}$$

By multiplying both inequalities (2.5) and (2.6), we get

$$\rho(\upsilon_n, \upsilon_{n+1})^{2(1-\alpha-\beta)} \leq \rho(\upsilon_{n-1}, \upsilon_n)^{\alpha+\beta} \tag{2.8}$$

Here, both inequalities (2.7) and (2.8) give

$$\rho(\upsilon_n, \upsilon_{n+1})^{2(1-\alpha-\beta)} = \rho(\upsilon_{n-1}, \upsilon_n)^{\alpha+\beta} \tag{2.9}$$

By adopting the relations (2.5)-(2.9) in inequality (2.3) and multiplying both sides by $\rho(\upsilon_n, \upsilon_{n+1})^{1-\alpha-\beta}$, we obtain the following:

$$\begin{aligned} \rho(\upsilon_n, \upsilon_{n+1})^{2-\alpha-\beta} \leq \lambda\big[&\rho(\upsilon_{n-1}, \upsilon_n)^{3\alpha}\rho(\upsilon_n, \upsilon_{n+1})^{1-\alpha-\beta} \\ &+ \rho(\upsilon_{n-1}, \upsilon_n)^{3\beta}\rho(\upsilon_n, \upsilon_{n+1})^{1-\alpha-\beta} + \rho(\upsilon_n, \upsilon_{n+1})^{4(1-\alpha-\beta)}\big]. \end{aligned} \tag{2.10}$$

This further reduces to

$$\begin{aligned}\rho(\upsilon_n,\upsilon_{n+1})^{2-\alpha-\beta} &\leq \lambda\left[\rho(\upsilon_{n-1},\upsilon_n)^{3\alpha}\rho(\upsilon_{n-1},\upsilon_n)^{2\beta-\alpha}+\rho(\upsilon_{n-1},\upsilon_n)^{3\beta}\rho(\upsilon_{n-1},\upsilon_n)^{2\alpha-\beta}\right.\\ &\qquad\left.+\rho(\upsilon_{n-1},\upsilon_n)^{2(\alpha+\beta)}\right]\\ &= \lambda\left[\rho(\upsilon_{n-1},\upsilon_n)^{2(\alpha+\beta)}+\rho(\upsilon_{n-1},\upsilon_n)^{2(\alpha+\beta)}+\rho(\upsilon_{n-1},\upsilon_n)^{2(\alpha+\beta)}\right]\\ &= 3\lambda\rho(\upsilon_{n-1},\upsilon_n)^{2(\alpha+\beta)} \equiv \mu\rho(\upsilon_{n-1},\upsilon_n)^{2(\alpha+\beta)}.\end{aligned}$$

Since $\alpha+\beta<1$, this last inequality gives

$$\rho(\upsilon_n,\upsilon_{n+1})^{2-\alpha-\beta} \leq \mu\rho(\upsilon_{n-1},\upsilon_n)^{\alpha+\beta}.$$

By applying equality (2.9), this further gives

$$\rho(\upsilon_n,\upsilon_{n+1}) \leq \mu\rho(\upsilon_{n-1},\upsilon_n). \tag{2.11}$$

By induction, inequality (2.11) becomes

$$\rho(\upsilon_n,\upsilon_{n+1}) \leq \mu^n\rho(\upsilon_0,\upsilon_1) \tag{2.12}$$

If $\rho(\upsilon_0,\upsilon_1)=0$, then $\upsilon_n=\upsilon_{n+1}$ for each $n=0,1,2,\ldots$. Thus, the proof is over. Now, let $\upsilon_n\neq\upsilon_k$ for all distinct $n,k\in\mathbb{N}$. Set $k=n+m$ for $m>1$, then by hypothesis on Ξ,

$$\begin{aligned}\rho(\upsilon_n,\upsilon_{n+m}) &\leq \rho(\upsilon_n,\upsilon_{n+1})+\rho(\upsilon_{n+1},\upsilon_{n+m})\\ &\vdots\\ &\leq \rho(\upsilon_n,\upsilon_{n+1})+\rho(\upsilon_{n+1},\upsilon_{n+2})+\cdots+\rho(\upsilon_{n+m-1},\upsilon_{n+m})\\ &= \sum_{i=0}^{m-1}\rho(\upsilon_{n+i},\upsilon_{n+i+1})\\ &\leq \sum_{i=0}^{m-1}\mu^{n+i}\rho(\upsilon_0,\upsilon_1)\end{aligned}$$

Since $d(p_0,p_1)\neq 0$, $\mu<1$, and taking limit as $n,m\to\infty$ across the last inequality, we have

$$\lim_{n\to\infty}\rho(\upsilon_n,\upsilon_{n+m})=0.$$

Hence, $\{\upsilon_n\}$ is a Cauchy sequence. By completeness of Ξ, $\{\upsilon_n\}$ converges to a limit υ^* (say) in Ξ, that is,

$$\lim_{n\to\infty}\upsilon_n=\upsilon^*.$$

Assume, on contrary, that υ^* is not a fixed point of Φ, we use the hypothesis on Ξ to get

$$\rho(\upsilon^*,\Phi\upsilon^*) \leq \rho(\upsilon^*,\upsilon_{n+1})+\rho(\upsilon_{n+1},\Phi\upsilon^*). \tag{2.13}$$

By condition (2.1), we have that

$$\begin{aligned}\rho(\upsilon_{n+1},\Phi\upsilon^*) &= \rho(\Phi\upsilon_n,\Phi\upsilon^*)\\ &\leq \lambda\left[\rho(\upsilon_n,\upsilon^*)^{3\alpha}+\rho(\upsilon_n,\Phi\upsilon_n)^{3\beta}+\rho(\upsilon^*,\Phi\upsilon^*)^{3(1-\alpha-\beta)}\right]\\ &= \lambda\left[\rho(\upsilon_n,\upsilon^*)^{3\alpha}+\rho(\upsilon_n,\upsilon_{n+1})^{3\beta}+\rho(\upsilon^*,\Phi\upsilon^*)^{3(1-\alpha-\beta)}\right]\end{aligned}$$

By adopting this last inequality into (2.13) and taking limit as $n \to \infty$, we obtain

$$\begin{aligned} &\rho(\upsilon^*, \Phi\upsilon^*) \leq \rho(\upsilon^*, \upsilon_{n+1}) + \lambda\rho(\upsilon^*, \Phi\upsilon^*)^{3(1-\alpha-\beta)} \\ &\Rightarrow \ \rho(\upsilon^*, \Phi\upsilon^*)^{3(\alpha+\beta)-2} \leq \lambda < \frac{1}{3}, \end{aligned}$$

which is a contradiction since $\alpha + \beta < 1$. Hence, $\upsilon^* = \Phi\upsilon^*$.
Finally, assume that υ^* and ω^* are two distinct fixed points of Φ, then

$$\begin{aligned} \rho(\upsilon^*, \omega^*) &= \rho(\Phi\upsilon^*, \Phi\omega^*) \\ &\leq \lambda \left[\rho(\upsilon^*, \omega^*)^{3\alpha} + \rho(\upsilon^*, \Phi\upsilon^*)^{3\beta} + \rho(\omega^*, \Phi\omega^*)^{3(1-\alpha-\beta)}\right] \\ &= \lambda\rho(\upsilon^*, \omega^*)^{3\alpha} \end{aligned}$$

That is,

$$\rho(\upsilon^*, \omega^*)^{1-3\alpha} \leq \lambda < \frac{1}{3}.$$

This leads to a contradiction. Thus, $\upsilon^* = \omega^*$. □

The following corollary is strictly included in Theorem 2.2.

Corollary 2.3. *Let (Ξ, ρ) be a complete metric space and $\Phi : \Xi \to \Xi$ is a map satisfying*

$$\rho(\Phi\upsilon, \Phi\omega) \begin{cases} \geq & \mu\sqrt[3]{\rho(\upsilon, \omega)\rho(\upsilon, \Phi\upsilon)\rho(\omega, \Phi\omega)} \\ \\ \leq & \lambda\left[\rho(\upsilon, \omega) + \rho(\upsilon, \Phi\upsilon) + \rho(\omega, \Phi\omega)\right] \end{cases}$$

where $\lambda \in (0, \frac{1}{3})$, $\mu \in (0, 1)$, for $\upsilon, \omega \in \Xi \setminus Fix(\Phi)$. Then, Φ has a unique fixed point in Ξ.

Proof. On putting $\alpha = \beta = \frac{1}{3}$, the proof follows from Theorem 2.2. □

Remark 2.4. (i) The result of Corollary 2.3 is a unification of both RRC map [10] and IRRC-type map [7].
(ii) If $\Xi \setminus \mathrm{Fix}(\Phi)$ is replaced with Ξ in Theorem 2.2, the following theorem would be suitable to solve both unique and non-unique fixed point problems.

Theorem 2.5. *Let (Ξ, ρ) be a complete metric space and $\Phi : \Xi \to \Xi$ be a map satisfying*

$$\rho(\Phi\upsilon, \Phi\omega) \leq \lambda\left[\rho(\upsilon, \omega)^{3\alpha} + \rho(\upsilon, \Phi\upsilon)^{3\beta} + \rho(\omega, \Phi\omega)^{3(1-\alpha-\beta)}\right] \tag{2.14}$$

for all $\upsilon, \omega \in \Xi$, $\lambda \in (0, \frac{1}{3})$, and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta < 1$. Then, Φ has a unique fixed point in Ξ.

Proof. Let $\upsilon_0 \in \Xi$ be a generic point and sequence $\{\upsilon_n\} \subset \Xi$, define $\upsilon_{n+1} = \Phi\upsilon_n$ for $n \geq 0$. Let $\upsilon_n \neq \upsilon_{n+1}$, $\upsilon = \upsilon_{n-1}$, and $\omega = \upsilon_n$ in (2.14), then

$$\begin{aligned} \rho(\upsilon_n, \upsilon_{n+1}) &= \rho(\Phi\upsilon_{n-1}, \Phi\upsilon_n) \\ &\leq \lambda\left[\rho(\upsilon_{n-1}, \upsilon_n)^{3\alpha} + \rho(\upsilon_{n-1}, \Phi\upsilon_{n-1})^{3\beta} + \rho(\upsilon_n, \Phi\upsilon_n)^{3(1-\alpha-\beta)}\right] \\ &= \lambda\left[\rho(\upsilon_{n-1}, \upsilon_n)^{3\alpha} + \rho(\upsilon_{n-1}, \upsilon_n)^{3\beta} + \rho(\upsilon_n, \upsilon_{n+1})^{3(1-\alpha-\beta)}\right] \qquad (2.15) \\ &\equiv \kappa\frac{\left[\rho(\upsilon_{n-1}, \upsilon_n)^{3\alpha} + \rho(\upsilon_{n-1}, \upsilon_n)^{3\beta} + \rho(\upsilon_n, \upsilon_{n+1})^{3(1-\alpha-\beta)}\right]}{3} \end{aligned}$$

where $\kappa \equiv 3\lambda < 1$. By taking the maximum of the left side of inequality (2.15) as $\rho(\upsilon_{n-1}, \upsilon_n)^{3\alpha}$, $\rho(\upsilon_{n-1}, \upsilon_n)^{3\beta}$, and $\rho(\upsilon_n, \upsilon_{n+1})^{3(1-\alpha-\beta)}$, we obtain the following three inequalities, respectively,

$$\rho(\upsilon_n, \upsilon_{n+1}) \leq \kappa\rho(\upsilon_{n-1}, \upsilon_n)^{3\alpha}, \tag{2.16}$$

$$\rho(\upsilon_n, \upsilon_{n+1}) \leq \kappa\rho(\upsilon_{n-1}, \upsilon_n)^{3\beta}, \tag{2.17}$$

and

$$\rho(\upsilon_n, \upsilon_{n+1}) \leq \kappa\rho(\upsilon_n, \upsilon_{n+1})^{3(1-\alpha-\beta)}. \tag{2.18}$$

By multiplying inequalities (2.16) - (2.18), we have

$$\rho(\upsilon_n, \upsilon_{n+1})^3 \leq \kappa^3\rho(\upsilon_{n-1}, \upsilon_n)^{3(\alpha+\beta)}\rho(\upsilon_n, \upsilon_{n+1})^{3(1-\alpha-\beta)}. \tag{2.19}$$

This further implies that

$$\rho(\upsilon_n, \upsilon_{n+1}) \leq \kappa^{\frac{1}{\alpha+\beta}}\rho(\upsilon_{n-1}, \upsilon_n).$$

Inductively, we have

$$\rho(\upsilon_n, \upsilon_{n+1}) \leq \prod_{j=0}^{n-1} \kappa^{(\frac{1}{\alpha+\beta})^{j+1}} \rho(\upsilon_0, \upsilon_1) \tag{2.20}$$

Obviously $\upsilon_n = \upsilon_{n+1}$, for each $n = 0, 1, 2, \ldots$, if $\rho(\upsilon_0, \upsilon_1) = 0$. Assume that $\upsilon_n \neq \upsilon_k$ for all distinct $n, k \in \mathbb{N}$. Set $k = n + m$ for $m > 1$, then by hypothesis on Ξ,

$$\begin{aligned}
\rho(\upsilon_n, \upsilon_{n+m}) &\leq \rho(\upsilon_n, \upsilon_{n+1}) + \rho(\upsilon_{n+1}, \upsilon_{n+m}) \\
&\vdots \\
&\leq \rho(\upsilon_n, \upsilon_{n+1}) + \rho(\upsilon_{n+1}, \upsilon_{n+2}) + \cdots + \rho(\upsilon_{n-m+1}, \upsilon_{n+m}) \\
&= \sum_{i=0}^{m-1} \rho(\upsilon_{n+i}, \upsilon_{n+i+1}) \\
&\leq \sum_{i=0}^{m-1} \left[\prod_{j=0}^{n+i-1} \kappa^{(\frac{1}{\alpha+\beta})^j} \rho(\upsilon_0, \upsilon_1) \right]
\end{aligned}$$

Since $d(p_0, p_1) \neq 0$, $\alpha + \beta < 1$, and $\kappa < 1$, then

$$\lim_{n\to\infty} \rho(\upsilon_n, \upsilon_{n+m}) = 0, \text{ as } n, m \to \infty$$

Therefore, $\{\upsilon_n\}$ is a Cauchy sequence. The rest of the proof is a routine. □

Corollary 2.6. *Let (Ξ, ρ) be a complete metric space and $\Phi : \Xi \to \Xi$ be a RRC-type contraction*

$$\rho(\Phi\upsilon, \Phi\omega) \leq \lambda \left[\rho(\upsilon, \omega) + \rho(\upsilon, \Phi\upsilon) + \rho(\omega, \Phi\omega)\right]$$

where $\lambda \in (0, \frac{1}{3})$ for $\upsilon, \omega \in \Xi$. Then, Φ has a unique fixed point in Ξ.

Proof. The proof is immediate if $\alpha = \beta = \frac{1}{3}$ in Theorem 2.5. □

We demonstrate the fulfillment of Theorems 2.2 and 2.5 using the following example.

Example 2.7. Let $\Xi = \{x_1, x_2, x_3, x_4\}$ and $\rho : \Xi^2 \to \mathbb{R}^+$ be the metric function given for all $(\upsilon, \omega) \in \Xi^2$ in Table 1. Also, let $\Phi : \Xi \to \Xi$ be the self-map

TABLE 1. Metric function ρ

$\rho(\upsilon, \omega)$	x_1	x_2	x_3	x_4
x_1	0	2	3	6
x_2	2	0	1	4
x_3	3	1	0	3
x_4	6	4	3	0

$$\Phi\upsilon = \begin{cases} x_1 & \text{if } \upsilon \in \{x_1, x_2, x_4\} \\ x_2 & \text{if } \upsilon \in \{x_3\} \end{cases}$$

In this example, we note that the pair (Ξ, ρ) is a metric space. More so, by defining a sequence $\{\upsilon_n\} \subset (\Xi, \rho)$ as

$$\upsilon_n = \begin{cases} x_1 \text{ if } n \leq 2, \\ x_2 \text{ if } 2 < n \leq 6, \\ x_4 \text{ if } n > 6. \end{cases}$$

Let $\mathcal{U} \subset (\Xi, \rho)$ be a topology of $\{\upsilon_n\}$, then $\{\upsilon_n\}$ is a Cauchy sequence since for every $m, n \in \mathbb{N}$ with $m > n$, there exists $u_m \in \mathcal{U}$, that is, $\rho(\upsilon_n, \upsilon_m) \to 0$ as $m, n \to \infty$. Furthermore, $\lim_{n\to\infty} \rho(\upsilon_n, x_4) \to 0$ as $n \to \infty$.
Next, for any $\upsilon, \omega \in \Xi \backslash Fix(\Phi)$, we select $\alpha = 0.25$, $\beta = 0.55$, $\mu = 0.8$ and $\lambda = 0.26$, then $\rho(\Phi\upsilon, \Phi\omega)$ lies between CRRC1 and CRRC2. Therefore, Φ satisfies the hypotheses of Theorem 2.2 with fixed point x_1.
Moreover, Φ fulfills the conditions of Theorem 2.5 for all $\upsilon, \omega \in \Xi$.
On the other hand, the map Φ does not satisfy the IRRC-type map, that is, by selecting $\upsilon = x_3$ and $\omega = x_4$,

$$\begin{aligned} \rho(\Phi x_3, \Phi x_4) &= \rho(x_2, x_1) \\ &= 2 \\ &> 1.5066 \\ &\approx \frac{4}{5}\rho(x_3, x_4)^{\alpha}\rho(x_3, x_2)^{\beta}\rho(x_4, x_1)^{1-\alpha-\beta} \\ &= \mu\rho(x_3, x_4)^{\alpha}\rho(x_3, \Phi x_3)^{\beta}\rho(x_4, \Phi x_4)^{1-\alpha-\beta} \end{aligned}$$

This implies that Φ fails to satisfy condition (1.5).

3. CONCLUSION

This study introduced a new CRRC-type technique to examine the existence of unique and non-unique fixed point problems. We proved some fixed point theorems of the new CRRC-type map, and all study hypotheses were validated using Example 2.7. The results showed that the CRRC-type is suitable to study the

existence of unique and non-unique fixed points when the IRRC-type map fails. Future studies would be tailored by determining the suitable constants α and β for approximating several fixed-point problems with or without emphasis on the contracting constants λ and μ.

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