

SOLUTIONS OF THE DIRAC EQUATION WITH THE SHIFTED DENG–FAN POTENTIAL INCLUDING YUKAWA-LIKE TENSOR INTERACTION

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By using the Nikiforov–Uvarov method, we give the approximate analytical solutions of the Dirac equation with the shifted Deng–Fan potential including the Yukawa-like tensor interaction under the spin and pseudospin symmetry conditions. After using an improved approximation scheme, we solved the resulting schrödinger-like equation analytically. Numerical results of the energy eigenvalues are also obtained, as expected, the tensor interaction removes degeneracies between spin and pseudospin doublets.

Keywords: Dirac equation; shifted Deng–Fan potential; Nikiforov–Uvarov method; spinors; Jacobi polynomials.

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1. Introduction

Pseudospin was introduced in the late 60s^{1,2} and it has been used to explain features of deformed nuclei³ and superdeformation,⁴ and to establish an effective shell-model coupling scheme.⁵ Spin and pseudospin symmetry now play an important role in Hadron and nuclear spectroscopy.^{6–8}

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In recent years, a lot of attention has been paid to Dirac equation with spin and pseudospin symmetry and their applications. For example, Zhou *et al.*⁹ used the relativistic mean field theory to investigate single antinucleon spectra. The spin symmetry in antinucleon spectra of a nucleus have been tested by investigating the relations between the Dirac wave functions of the spin doublets, and these relations in realistic nuclei within the relativistic mean-field model have been examined.¹⁰ From the Dirac equation, the mechanism behind the pseudospin symmetry was studied by Meng *et al.*^{11,12} and the pseudospin symmetry was shown to be connected with the competition between the centrifugal barrier and the pseudospin orbital potential, which is mainly decided by the derivative of the difference between the scalar and vector potentials.

Lalazissis *et al.*¹³ investigated the pseudospin approximation in exotic nuclei in Zr and Sn isotopes from the proton drip line to the neutron drip line based on the relativistic continuum Hartree–Bogoliubov theory. Nuclear halo structure and conservation of relativistic symmetry have been studied within the framework of the relativistic Hartree–Fock–Bogoliubov theory by Long *et al.*¹⁴

Very recently, Song *et al.*¹⁵ studied the effects of tensor coupling on the spin symmetry of $\bar{\Lambda}$ spectra in $\bar{\Lambda}$ -nucleus systems with the relativistic mean-field theory. Shortly thereafter, Lu *et al.*¹⁶ show that the conservation and the breaking of the pseudospin symmetry in resonant states and bound states share some similar properties. Furthermore in 2013, by examining the zeros of Jost functions corresponding to the small components of Dirac wave functions and phase shifts of continuum states, Lu *et al.*¹⁶ also showed that the pseudospin symmetry in single particle resonant states in nuclei is conserved when the attractive scalar and repulsive vector potentials have the same magnitude but opposite sign.¹⁷ Li *et al.*¹⁸ proved that the spin-orbit interactions always play a role in favor of the pseudospin symmetry, and whether the pseudospin symmetry is improved or destroyed by the dynamical term relating the shape of the potential as well as the quantum numbers of the state.

In the recent years, some authors have investigated the spin symmetry and pseudospin symmetry with the Dirac equation for some typical potentials such as the Harmonic oscillator potential,^{19–27} Coulomb potential,²⁸ Woods–Saxon potential,^{29,30} Morse potential,^{31–34} Eckart potential,^{35,36} ring-shaped nonspherical harmonic oscillator,³⁷ Pöschl–Teller potential,^{38–42} three parameter potential function as a diatomic molecule model,⁴³ Yukawa potential.^{44–46} Others are pseudoharmonic potential,⁴⁷ Davidson potential,⁴⁸ Mie-type potential,⁴⁹ Deng–Fan potential,⁵⁰ hyperbolic potential,⁵¹ Tietz potential⁵² and Rosen–Morse potential.^{53–55}

Different methods have been used such as supersymmetric quantum mechanics, Nikiforov–Uvarov (NU) method, path integral, series expansion and ansatz techniques. In this paper, we have used the elegant NU method to obtain the improved approximate analytical solutions of the Dirac equation under the shifted Deng–Fan scalar and vector potentials and a Yukawa-like tensor. The improved approximate analytical solutions can be obtained by using the improved new approximation scheme (Pekeris-type approximation scheme, Pekeris⁵⁶) to deal with

the centrifugal (pseudo-centrifugal) term near the minimum point $r = r_e$.⁵⁷ In addition, the corresponding lower and upper component spinors are obtained.

The Deng-Fan molecular potential, which is an exponential-type potential, is a simple modified Morse potential called generalized Morse potential which was proposed by Deng and Fan in 1957,⁵⁸ in an attempt to find a more suitable diatomic potential to describe the vibrational spectrum.⁵⁹ This potential has been widely studied by some researchers in various applications.^{57–63} This potential model can be used to describe the motion of the nucleons in the mean field produced by the interactions between nuclei.⁵⁷ Maghsoodi *et al.* obtained the spectrum of Dirac equation under the Deng-Fan potential with a tensor interaction, using supersymmetric quantum mechanics.⁵⁰

Very recently, Hamzavi *et al.*⁶⁴ suggested a modified form of the Deng-Fan potential, called the shifted Deng-Fan potential. The modification is the Deng-Fan potential shifted by dissociation energy $\bar{\alpha}$.⁶⁴ In their work, it was demonstrated that shifted Deng-Fan potential and the Morse potential are very close to each other for large values of r in the regions $r \approx r_e$ and $r > r_e$, but different at $r \approx 0$.⁶⁴ Also, the thermodynamic properties and the approximate solutions of the Schrödinger equation with this potential model have been studied by Oyewumi *et al.*⁶⁵

In order to obtain theoretical results that are comparable with the experimental results, a tensor interaction is always been introduced in the Dirac equation with the pseudospin and spin symmetry limits to remove the degeneracy in the pseudospin and spin doublets.^{25–27,66–69} Tensor couplings or interactions have been used successfully in the studies of nuclear properties and applications.^{8,23,25–28,34,40,44–47,49–52,55,66–72,74–79}

In most of these investigations, Coulomb-like tensor interaction has been used, except in a few instances where linear plus Coulomb tensor interaction has been used.^{40,74} Recently, another form of tensor interaction (Yukawa potential as a tensor interaction) has been introduced by Hassanabadi *et al.*⁶⁹ and Aydoğdu *et al.*⁷⁹

It is therefore the purpose of the present study to investigate the Dirac equation with the shifted Deng-Fan potential coupled with the Yukawa tensor potential (instead of the usual Coulomb-like tensor) under the pseudospin and spin symmetry limits. The approximate scheme (the Pekeris-type approximation scheme) has been used to deal with the (pseudo-) centrifugal term, this problem has been solved via the NU method.

The scheme of our presentations are as follows: In Sec. 2, we give a brief review of the Dirac equation with Yukawa tensor interaction. Section 3 contains the approximate solution to the problem. The conclusions and some numerical results for the energy spectrum are given in Sec. 4.

2. The Dirac Bound State Solutions Including Yukawa-Like Tensor

The time-independent Dirac equation with the scalar potential $S(r)$, vector potential $V(r)$ and a tensor potential $U(r)$ is written for a spin-half particle in

the natural units $\hbar = c = 1$ as^{6,21,46,50,54,55,80}

$$[\alpha \cdot \mathbf{p} + \beta(M + S(r)) - i\beta\alpha \cdot \hat{r}U(r)]\Psi_{n\kappa}(\mathbf{r}) = [E - V(r)]\Psi_{n\kappa}(\mathbf{r}), \quad (1)$$

where E and M are the relativistic energy of the system and the rest mass of the spin- $\frac{1}{2}$ particle, respectively. $\mathbf{p} = -i\nabla$ is the three-dimensional momentum operator. α and β are 4×4 Dirac matrices, given as

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (2)$$

where I is the 2×2 identity matrix and σ_i ($i = 1, 2, 3$) are the three 2×2 Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

For a particle in a spherical (central) field, the total angular momentum operator $\mathbf{J}$ and the spin-orbit matrix operator $\hat{\mathbf{K}} = -\beta(\sigma \cdot \mathbf{L} + \mathbf{I})$ commute with the Dirac Hamiltonian, where $\mathbf{L}$ is the orbital angular momentum operator. For a given total angular momentum j , the eigenvalues of $\hat{\mathbf{K}}$ are $\kappa = -(j + \frac{1}{2})$ for aligned spin ($s_{\frac{1}{2}}, p_{\frac{3}{2}}$, etc) and $\kappa = j + \frac{1}{2}$ for unaligned spin ($p_{\frac{1}{2}}, d_{\frac{3}{2}}$, etc). The spinor wave functions can be classified according to the radial quantum number, n and the spin-orbit quantum number κ and can be written using the Pauli-Dirac representation^{6,7,21}:

$$\Psi_{n\kappa}(\mathbf{r}) = \frac{1}{r} \begin{bmatrix} F_{n\kappa}(\mathbf{r})Y_{jm}^\ell(\theta, \vartheta) \\ iG_{n\kappa}(\mathbf{r})Y_{jm}^{\tilde{\ell}}(\theta, \vartheta) \end{bmatrix}, \quad (4)$$

where $F_{n\kappa}(\mathbf{r})$ and $G_{n\kappa}(\mathbf{r})$ are the radial wave functions of the upper and lower spinor components, respectively. $Y_{jm}^\ell(\theta, \vartheta)$ and $Y_{jm}^{\tilde{\ell}}(\theta, \vartheta)$ are the spherical harmonics functions coupled to the total angular momentum j and its projection m on the z -axis. The orbital and pseudo-orbital angular momentum quantum numbers for spin symmetry ℓ and pseudo-spin symmetry $\tilde{\ell}$ refer to the upper and lower components, respectively.

Substituting Eq. (4) into Eq. (1), we obtain the two coupled first-order differential equations:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))G_{n\kappa}(r), \quad (5)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))F_{n\kappa}(r), \quad (6)$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$ are the difference and the sum potentials, respectively. By eliminating $G_{n\kappa}(r)$ in Eq. (5) and $F_{n\kappa}(r)$ in Eq. (6), we obtain two second-order nonlinear differential equations for the upper and lower

radial spinor components, respectively, as

$$\begin{aligned} & \left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{dU(r)}{dr} - U^2(r) + \frac{\frac{d\Delta(r)}{dr}}{M + E_{n\kappa} - \Delta(r)} \right. \\ & \quad \times \left. \left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) \right] F_{n\kappa}(r) \\ & = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))] F_{n\kappa}(r), \end{aligned} \quad (7)$$

$$\begin{aligned} & \left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{dU(r)}{dr} - U^2(r) + \frac{\frac{d\Sigma(r)}{dr}}{M - E_{n\kappa} + \Sigma(r)} \right. \\ & \quad \times \left. \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) \right] G_{n\kappa}(r) \\ & = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))] G_{n\kappa}(r), \end{aligned} \quad (8)$$

where $\kappa(\kappa-1) = \tilde{\ell}(\tilde{\ell}+1)$ and $\kappa(\kappa+1) = \ell(\ell+1)$.

2.1. Spin symmetry limit

The spin symmetry occurs when $\frac{d\Delta(r)}{dr} = 0$ or $\Delta(r) = C_s = \text{constant}$.^{6,7,21,36,50,57} Now taking $\Sigma(r)$ as the shifted Deng-Fan potential,⁶⁴ i.e.,

$$\Sigma(r) = Db \left[\frac{b}{(e^{ar} - 1)^2} - \frac{2}{e^{ar} - 1} \right], \quad b = e^{ar_e} - 1, \quad (9)$$

where $r \in (0, \infty)$, the three positive parameters D , r_e and a denote the dissociation energy, the equilibrium inter-nuclear distance and the range of the potential well, respectively. The Yukawa tensor potential is introduced instead of the usual Coulomb-like tensor interaction as

$$U(r) = -\frac{A}{r}e^{-ar}, \quad (10)$$

where $A = \gamma Z$, $\gamma = (137.037)^{-1}$ is the fine-structure constant, Z is the atomic number. This potential is often used to compute the bound-state normalizations and energy levels of neutral atoms.⁴⁶ Under this symmetry, Eq. (7) can easily be transformed into the following equation:

$$\begin{aligned} & \left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1) + (2\kappa+1)Ae^{-ar} + A^2e^{-2ar}}{r^2} - \frac{Aae^{-ar}}{r} \right] F_{s,n\kappa}(r) \\ & \quad \times \left[(M + E_{n\kappa} - C_s) \left(M - E_{n\kappa} + Db \left(\frac{b}{(e^{ar} - 1)^2} - \frac{2}{e^{ar} - 1} \right) \right) \right] = 0, \end{aligned} \quad (11)$$

where $\kappa = \ell$ and $\kappa = -\ell - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

2.2. Pseudospin symmetry limit

The pseudospin symmetry occurs when $\frac{d[V(r)+S(r)]}{dr} = \frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = C_{\text{ps}} = \text{constant}$. Here we are taking $\Delta(r)$ as the shifted Deng–Fan potential and the tensor as the Yukawa-like potential, i.e.,

$$\Delta(r) = Db \left[\frac{b}{(e^{ar} - 1)^2} - \frac{2}{e^{ar} - 1} \right], \quad U(r) = -\frac{A}{r}e^{-ar}, \quad r \geq R_c. \quad (12)$$

By substituting Eq. (12) into Eq. (8), we can easily obtain

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1) + (2\kappa + 1)Ae^{-ar} + A^2e^{-2ar}}{r^2} - \frac{Aae^{-ar}}{r} \right] G_{ps,n\kappa}(r) \\ \times \left[(M - E_{n\kappa} + C_{\text{ps}}) \left(M + E_{n\kappa} - Db \left(\frac{b}{(e^{ar} - 1)^2} - \frac{2}{e^{ar} - 1} \right) \right) \right] = 0, \quad (13)$$

where $\kappa = -\tilde{\ell}$ and $\kappa = \tilde{\ell} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

3. Bound State Solutions of the Dirac Equation with the Shifted Deng–Fan Potential and the Yukawa Tensor Potential

In this section, by employing the NU method, we shall approximately solve the Dirac equation with the shifted Deng–Fan potential and the Yukawa tensor potential.

3.1. Solutions of spin symmetry limit

In order to obtain the exact solution of Eq. (11) for $\kappa \neq (0, -1)$, we must therefore include an approximation to deal with the centrifugal barrier. It is found that for short-range potential, the following approximation is a good approximation to the spin/pseudospin orbit coupling term:

$$\frac{1}{r^2} \approx \frac{a^2}{(1 - e^{-ar})^2}. \quad (14)$$

In order to show the validity of the approximation in Eq. (14), we define the following functions which are similar to the works of Dong⁶⁰ and Falaye⁸²:

$$f_1(r) = \frac{1}{r^2}, \quad f_2(r) = \frac{a^2}{(1 - e^{-ar})^2}, \quad a = 0.1, 0.5, 1.0, \quad (15)$$

and show the plots of expressions $f_1(r)$ and $f_2(r)$ as a function of the variable r with parameter $a = 0.1, 0.5, 1.0$ in Fig. 1. It is shown that when a is small (short potential range), Eq. (14) is a good approximation to $\frac{1}{r^2}$. However, the difference becomes noticeable for large a (long-range potential). This implies that the Eq. (14) is not a good approximation to the centrifugal term when the parameter a becomes large.

Now, with the aid of Eq. (14) and by introducing the following new dimensionless parameter, $z(r) = e^{-ar} \in [0, 1]$, which maintains the finiteness of the transformed

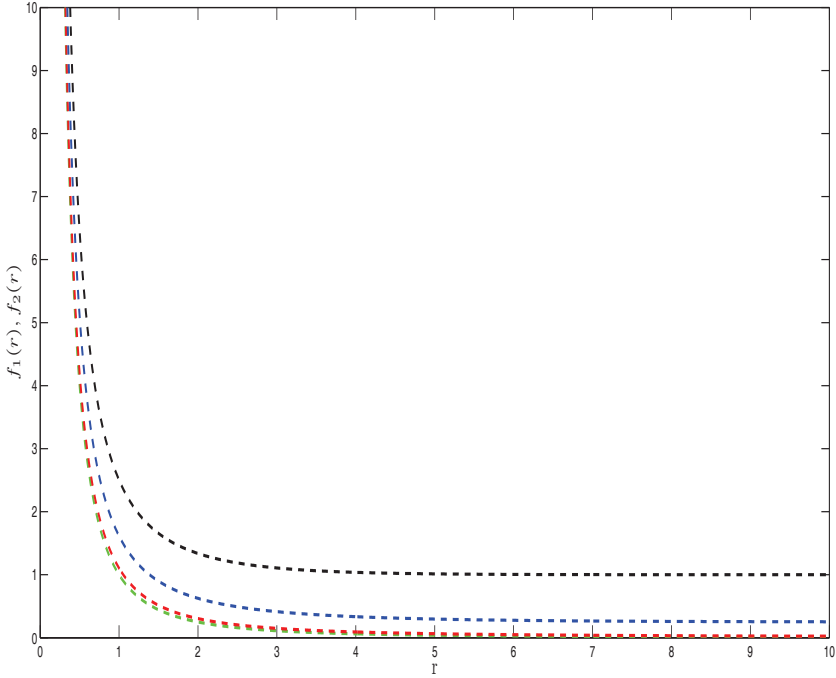


Fig. 1. (Color online) The plots of expressions $f_1(r) = \frac{1}{r^2}$ (green dash line) and $f_2(r) = \frac{a^2}{(1-e^{-ar})^2}$ as a function of the variable r are displayed with parameters $a = 0.1, 0.5, 1.0$ correspond to red dash, blue dash and black dash lines, respectively.

wave functions on the boundary conditions, Eq. (11) is easily transformed into the following Schrödinger-like equation satisfying $F_{s,n\kappa}(r)$:

$$\begin{aligned} & \frac{d^2 F_{s,n\kappa}}{dz^2} + \frac{1}{z} \frac{dF_{s,n\kappa}}{dz} \\ & - \left\{ \frac{(A^2 - A + \xi(b+2) + \zeta) z^2 + (2A(\kappa+1) - 2\xi - 2\zeta) z + \kappa(\kappa+1) + \zeta}{z^2(1-z)^2} \right\} \\ & \times F_{s,n\kappa}(z) = 0, \end{aligned} \quad (16)$$

where we have introduced the following parameters:

$$\xi = \frac{1}{a^2} (M + E_{n\kappa} - C_s) Db, \quad \zeta = \frac{1}{a^2} (M + E_{n\kappa} - C_s) (M - E_{n\kappa}). \quad (17)$$

Equation (16) can now be solved by using the NU method, and thus, the required polynomials given in Appendix A take the following analytic forms:

$$\pi(z) = -\frac{z}{2} - \left[\left[\sqrt{\left(A + \kappa + \frac{1}{2} \right)^2 + \xi b + \sqrt{\kappa(\kappa+1) + \zeta}} \right] z - \sqrt{\kappa(\kappa+1) + \zeta} \right], \quad (18a)$$

$$k = 2\xi - 2(A + \kappa)(k + 1) - 2\sqrt{(\kappa(\kappa + 1) + \zeta) \left[\left(A + \kappa + \frac{1}{2}\right)^2 + \xi b \right]}, \quad (18b)$$

$$\tau(z) = 1 - 2z - 2 \left[\left[\sqrt{\left(A + \kappa + \frac{1}{2}\right)^2 + \xi b} + \sqrt{\kappa(\kappa + 1) + \zeta} \right] z - \sqrt{\kappa(\kappa + 1) + \zeta} \right], \quad (18c)$$

with $\tau'(z) = -2 - 2\left[\sqrt{\left(A + \kappa + \frac{1}{2}\right)^2 + \xi b} + \sqrt{\kappa(\kappa + 1) + \zeta}\right] < 0$. It is straightforward to establish the following relation by using (A.10):

$$\left(n + \frac{1}{2} + \sqrt{\left(A + \kappa + \frac{1}{2}\right)^2 + \xi b} + \sqrt{\kappa(\kappa + 1) + \zeta}\right)^2 - A^2 + A - \xi(b + 2) - \zeta = 0. \quad (19)$$

Using the identities in Eq. (17), we can find a more explicit expression for the relativistic bound-state energy spectrum as

$$\begin{aligned} & \left(n + \frac{1}{2} + \sqrt{\left(A + \kappa + \frac{1}{2}\right)^2 + \frac{Db^2}{a^2}(M + E_{n\kappa} - C_s)} \right. \\ & \quad \left. + \sqrt{\kappa(\kappa + 1) + \frac{1}{a^2}(M + E_{n\kappa} - C_s)(M - E_{n\kappa})}\right)^2 \\ & \quad - A^2 + A - \frac{Db(b + 2)}{a^2}(M + E_{n\kappa} - C_s) - \frac{1}{a^2}(M + E_{n\kappa} - C_s)(M - E_{n\kappa}) = 0. \end{aligned} \quad (20)$$

In order to obtain the corresponding wave functions, we find the explicit form of the weight function (Appendix (A.10)) as

$$\rho(z) = z^{2\delta}(1 - z)^\eta, \quad (21)$$

$$\delta = \sqrt{\kappa(\kappa + 1) + \zeta}, \quad \eta = 2\sqrt{\left(A + \kappa + \frac{1}{2}\right)^2 + \frac{Db^2}{a^2}(M + E_{n\kappa} - C_s)}.$$

The above weight function gives the first part of the wave functions from (Appendix (A.9)) as

$$y_{n\kappa}(z) \rightarrow P_n^{(2\delta, \eta)}(1 - 2z) \quad (22)$$

and hence the second part of the wave functions can be easily found by using Appendix (A.5) as

$$\phi(z) \rightarrow z^\delta(1 - z)^{\frac{\eta}{2} + \frac{1}{2}}. \quad (23)$$

Hence, the unnormalized upper component of the spinor wave functions is expressed in terms of the Jacobi polynomials as (Appendix (A.2))

$$F_{s,n\kappa}(r) = N_{s,n\kappa} z^{\delta} (1-z)^{\frac{\eta}{2} + \frac{1}{2}} P_n^{(2\delta, \eta)}(1-2z), \quad (24)$$

where $N_{s,n\kappa}$ is the normalization constant.

3.2. Solutions of pseudospin symmetry limit

The lower component of the spinor wave function $G_{ps,n\kappa}$ in Eq. (13) can be obtained by following the same procedure in the previous subsection, this equation can be expressed as

$$\begin{aligned} & \frac{d^2 G_{ps,n\kappa}(z)}{dz^2} + \frac{1}{z} \frac{dG_{ps,n\kappa}(z)}{dz} \\ & - \left\{ \frac{\left(A^2 - A + \tilde{\xi}(b+2) + \tilde{\zeta} \right) z^2 + \left(2A(\kappa+1) - 2\tilde{\xi} - 2\tilde{\zeta} \right) z + \kappa(\kappa-1) + \tilde{\zeta}}{z^2(1-z)^2} \right\} \\ & \times G_{ps,n\kappa}(z) = 0, \end{aligned} \quad (25)$$

where the following parameters have been introduced:

$$\tilde{\xi} = \frac{1}{a^2} (E_{n\kappa} - M - C_s) Db, \quad \tilde{\zeta} = \frac{1}{a^2} (M - E_{n\kappa} + C_s)(M + E_{n\kappa}). \quad (26)$$

To avoid repetition in the procedure of obtaining the solution for this case, the following parameter mapping is used⁸¹:

$$\begin{aligned} F_{s,n\kappa}(r) & \leftrightarrow G_{ps,n\kappa}, \quad \kappa \rightarrow \kappa - 1, \quad V(r) \rightarrow -V(r), \\ E_{s,n\kappa} & \rightarrow -E_{ps,n\kappa}, \quad C_s \rightarrow -C_{ps}. \end{aligned} \quad (27)$$

By applying the above transformations, Eqs. (20), (24) and (25) lead to the following pseudospin symmetric energy equations:

$$\begin{aligned} & \left(n + \frac{1}{2} + \sqrt{\left(A + \kappa - \frac{1}{2} \right)^2 + \frac{Db^2}{a^2} (E_{n\kappa} - M - C_s)} \right. \\ & \quad \left. + \sqrt{\kappa(\kappa-1) + \frac{1}{a^2} (M - E_{n\kappa} + C_s)(M + E_{n\kappa})} \right)^2 \\ & - A^2 + A - \frac{Db(b+2)}{a^2} (E_{n\kappa} - M - C_s) - \frac{1}{a^2} (M - E_{n\kappa} + C_s)(M + E_{n\kappa}) = 0, \end{aligned} \quad (28)$$

and the unnormalized lower component of the spinor wave function is obtained in terms of the Jacobi polynomials as

$$G_{ps,n\kappa}(r) = \tilde{N}_{s,n\kappa} z^{\tilde{\delta}} (1-z)^{\frac{\tilde{\eta}}{2} + \frac{1}{2}} P_n^{(2\tilde{\delta}, \tilde{\eta})}(1-2z), \quad (29)$$

where $\tilde{N}_{s,n\kappa}$ is the normalization constant.

Table 1. The energies (in units fm^{-1}) in the spin symmetry limit for $M = 1 \text{ fm}^{-1}$, $D = 15$, $a = 0.1$, $C_s = 0.5$, $r_e = 0.8$, 0.4 fm^{-1} in atomic units $\hbar = c = \mu = 1$.

ℓ	$n, \kappa < 0$	$nL_{j=\ell+1/2}$	$E_{n,\kappa < 0} (A = 0)$	$n, \kappa > 0$	$nL_{j=\ell-1/2}$	$E_{n,\kappa > 0} (A = 0.5)$	$n, \kappa > 0$	$nL_{j=\ell-1/2}$	$E_{n,\kappa > 0} (A = 0.5)$
$C_s = 0, r_e = 0.8$									
1	0, -2	$0p_{3/2}$	-0.994680673675	0, 1	$0p_{1/2}$	-0.99513873187	0, 1	$0p_{1/2}$	-0.993888275050
2	0, -3	$0d_{5/2}$	-0.987388385545	0, 2	$0d_{3/2}$	-0.988323299165	0, 2	$0d_{3/2}$	-0.986225745520
3	0, -4	$0f_{5/2}$	-0.977137980190	0, 3	$0f_{3/2}$	-0.97836121346	0, 3	$0f_{3/2}$	-0.975605107885
4	0, -5	$0g_{9/2}$	-0.963933376195	0, 4	$0g_{7/2}$	-0.9652593054	0, 4	$0g_{7/2}$	-0.962032400570
1	1, -2	$1p_{3/2}$	-0.987392704065	1, 1	$1p_{1/2}$	-0.98792328953	1, 1	$1p_{1/2}$	-0.986459755860
2	1, -3	$1d_{5/2}$	-0.977723165240	1, 2	$1d_{3/2}$	-0.97866241436	1, 2	$1d_{3/2}$	-0.976418440970
3	1, -4	$1f_{7/2}$	-0.965030658185	1, 3	$1f_{5/2}$	-0.96636085862	1, 3	$1f_{5/2}$	-0.963356764465
4	1, -5	$1g_{9/2}$	-0.949378933000	1, 4	$1g_{7/2}$	-0.95108754925	1, 4	$1g_{7/2}$	-0.947339237365
$C_s = 0, r_e = 0.4$									
1	0, -2	$0p_{3/2}$	-0.994575000815	0, 1	$0p_{1/2}$	-0.995056043500	0, 1	$0p_{1/2}$	-0.993745845455
2	0, -3	$0d_{5/2}$	-0.987132159215	0, 2	$0d_{3/2}$	-0.988017930695	0, 2	$0d_{3/2}$	-0.985915834920
3	0, -4	$0f_{5/2}$	-0.976667194300	0, 3	$0f_{3/2}$	-0.977948605220	0, 3	$0f_{3/2}$	-0.975063776210
4	0, -5	$0g_{9/2}$	-0.963184734430	0, 4	$0g_{7/2}$	-0.964856450845	0, 4	$0g_{7/2}$	-0.961196532235
1	1, -2	$1p_{3/2}$	-0.987167626585	1, 1	$1p_{1/2}$	-0.987726126225	1, 1	$1p_{1/2}$	-0.986189096180
2	1, -3	$1d_{5/2}$	-0.977312285560	1, 2	$1d_{3/2}$	-0.978300677660	1, 2	$1d_{3/2}$	-0.975944854180
3	1, -4	$1f_{7/2}$	-0.964368211980	1, 3	$1f_{5/2}$	-0.965763718230	1, 3	$1f_{5/2}$	-0.962614829910
4	1, -5	$1g_{9/2}$	-0.948401747670	1, 4	$1g_{7/2}$	-0.950192903485	1, 4	$1g_{7/2}$	-0.946266072595
$C_s = 5, r_e = 0.8$									
1	0, -2	$0p_{3/2}$	4.00317053373	0, 1	$0p_{1/2}$	4.0029287387	0, 1	$0p_{1/2}$	4.00357380957
2	0, -3	$0d_{5/2}$	4.00804249641	0, 2	$0d_{3/2}$	4.00756296838	0, 2	$0d_{3/2}$	4.00868886968
3	0, -4	$0f_{5/2}$	4.01497524047	0, 3	$0f_{3/2}$	4.01426159363	0, 3	$0f_{3/2}$	4.01585663245
4	0, -5	$0g_{9/2}$	4.02397586139	0, 4	$0g_{7/2}$	4.02303016318	0, 4	$0g_{7/2}$	4.02508933373
1	1, -2	$1p_{3/2}$	4.00540551251	1, 1	$1p_{1/2}$	4.0052225504	1, 1	$1p_{1/2}$	4.00570704969
2	1, -3	$1d_{5/2}$	4.01185375973	1, 2	$1d_{3/2}$	4.01141639278	1, 2	$1d_{3/2}$	4.01244650857
3	1, -4	$1f_{7/2}$	4.02020368469	1, 3	$1f_{5/2}$	4.01952508055	1, 3	$1f_{5/2}$	4.02104493729
4	1, -5	$1g_{9/2}$	4.03057429824	1, 4	$1g_{7/2}$	4.02965950438	1, 4	$1g_{7/2}$	4.03165408392
$C_s = 5, r_e = 0.4$									
1	0, -2	$0p_{3/2}$	4.00322054764	0, 1	$0p_{1/2}$	4.00296888842	0, 1	$0p_{1/2}$	4.00363806577
2	0, -3	$0d_{5/2}$	4.00817915484	0, 2	$0d_{3/2}$	4.00768038478	0, 2	$0d_{3/2}$	4.00884949119
3	0, -4	$0f_{5/2}$	4.01523807128	0, 3	$0f_{3/2}$	4.01449607769	0, 3	$0f_{3/2}$	4.01615259373
4	0, -5	$0g_{9/2}$	4.02440431783	0, 4	$0g_{7/2}$	4.02342131161	0, 4	$0g_{7/2}$	4.02555988856
1	1, -2	$1p_{3/2}$	4.00545796030	1, 1	$1p_{1/2}$	4.00526892236	1, 1	$1p_{1/2}$	4.00576677707
2	1, -3	$1d_{5/2}$	4.01201064991	1, 2	$1d_{3/2}$	4.01155631255	1, 2	$1d_{3/2}$	4.0126243621
3	1, -4	$1f_{7/2}$	4.02050032447	1, 3	$1f_{5/2}$	4.01979510400	1, 3	$1f_{5/2}$	4.02137262058
4	1, -5	$1g_{9/2}$	4.03104902087	1, 4	$1g_{7/2}$	4.03009835517	1, 4	$1g_{7/2}$	4.03216924169

Table 2. The energies (in units fm^{-1}) in the pseudospin symmetry limit for $M = 1 \text{ fm}^{-1}$, $D = 15$, $a = 0.1$, $C_s = 0$, -5 , $r_e = 0.4$, 0.8 fm^{-1} in atomic units $\hbar = c = \mu = 1$.

ℓ	$n, \kappa < 0$	ℓ, j	$E_{n, \kappa < 0}$ ($A = 0$)	$n - 1, \kappa > 0$	$\ell + 2, j + 1$	$E_{n-1, \kappa > 0}$ ($A = 0$)	$n, \kappa < 0$	ℓ, j	$E_{n, \kappa < 0}$ ($A = 0.5$)	$n - 1, \kappa > 0$	$\ell + 2, j + 1$	$E_{n-1, \kappa > 0}$ ($A = 0.5$)
$C_{\text{ps}} = 0, r_e = 0.8$												
1	1, -1	1s _{1/2}	1.00634753849	0, 2	0d _{3/2}	1.00634753849	1, -1	1s _{1/2}	1.00610639381	0, 2	0d _{3/2}	1.00675977512
2	1, -2	1p _{3/2}	1.01327511440	0, 3	0f _{5/2}	1.01327511440	1, -2	1p _{3/2}	1.01276147189	0, 3	0f _{5/2}	1.0139720951
3	1, -3	1d _{5/2}	1.02227632704	0, 4	0g _{7/2}	1.02227632704	1, -3	1d _{5/2}	1.02149997780	0, 4	0g _{7/2}	1.02324344461
4	1, -4	1f _{7/2}	1.03344083138	0, 5	0h _{9/2}	1.03344083138	1, -4	1f _{7/2}	1.03240678672	0, 5	0h _{9/2}	1.03466554547
1	2, -1	2s _{1/2}	1.00919092054	1, 2	1d _{3/2}	1.00919092054	2, -1	2s _{1/2}	1.00903376458	1, 2	1d _{3/2}	1.00944522248
2	2, -2	2p _{3/2}	1.01837933133	1, 3	1f _{5/2}	1.01837933133	2, -2	2p _{3/2}	1.01790936762	1, 3	1f _{5/2}	1.01902421126
3	2, -3	2d _{5/2}	1.02909646038	1, 4	1g _{7/2}	1.02909646038	2, -3	2d _{5/2}	1.02835323755	1, 4	1g _{7/2}	1.0300254518
4	2, -4	2f _{7/2}	1.04185296565	1, 5	1h _{9/2}	1.04185296565	2, -4	2f _{7/2}	1.04084597013	1, 5	1h _{9/2}	1.04304855818
$C_{\text{ps}} = 0, r_e = 0.4$												
1	1, -1	1s _{1/2}	1.00642272478	0, 2	0d _{3/2}	1.00642272478	1, -1	1s _{1/2}	1.00617161853	0, 2	0d _{3/2}	1.00684974812
2	1, -2	1p _{3/2}	1.01346626112	0, 3	0f _{5/2}	1.01346626112	1, -2	1p _{3/2}	1.01293096354	0, 3	0f _{5/2}	1.01419592001
3	1, -3	1d _{5/2}	1.02262388110	0, 4	0g _{7/2}	1.02262388110	1, -3	1d _{5/2}	1.02181504291	0, 4	0g _{7/2}	1.02362950439
4	1, -4	1f _{7/2}	1.03398700259	0, 5	0h _{9/2}	1.03398700259	1, -4	1f _{7/2}	1.03291001560	0, 5	0h _{9/2}	1.03526065500
1	2, -1	2s _{1/2}	1.00925346428	1, 2	1d _{3/2}	1.00925346428	2, -1	2s _{1/2}	1.00909303459	1, 2	1d _{3/2}	1.00950917764
2	2, -2	2p _{3/2}	1.01859755340	1, 3	1f _{5/2}	1.01859755340	2, -2	2p _{3/2}	1.01810852148	1, 3	1f _{5/2}	1.01926660631
3	2, -3	2d _{5/2}	1.02949160803	1, 4	1g _{7/2}	1.02949160803	2, -3	2d _{5/2}	1.02871759605	1, 4	1g _{7/2}	1.03045714342
4	2, -4	2f _{7/2}	1.04246400301	1, 5	1h _{9/2}	1.04246400301	2, -4	2f _{7/2}	1.04141529809	1, 5	1h _{9/2}	1.04370719456
$C_{\text{ps}} = -5, r_e = 0.8$												
1	1, -1	1s _{1/2}	-3.98474235212	0, 2	0d _{3/2}	-3.98474235212	1, -1	1s _{1/2}	-3.98536953322	0, 2	0d _{3/2}	-3.98363892498
2	1, -2	1p _{3/2}	-3.97408332396	0, 3	0f _{5/2}	-3.97408332396	1, -2	1p _{3/2}	-3.97517771342	0, 3	0f _{5/2}	-3.97256443759
3	1, -3	1d _{5/2}	-3.96005901271	0, 4	0g _{7/2}	-3.96005901271	1, -3	1d _{5/2}	-3.96158951648	0, 4	0g _{7/2}	-3.9581305164
4	1, -4	1f _{7/2}	-3.94279155689	0, 5	0h _{9/2}	-3.94279155689	1, -4	1f _{7/2}	-3.94474554082	0, 5	0h _{9/2}	-3.94045911429
1	2, -1	2s _{1/2}	-3.96931123016	1, 2	1d _{3/2}	-3.96931123016	2, -1	2s _{1/2}	-3.96999215207	1, 2	1d _{3/2}	-3.96810323038
2	2, -2	2p _{3/2}	-3.95679768446	1, 3	1f _{5/2}	-3.95679768446	2, -2	2p _{3/2}	-3.95798950402	1, 3	1f _{5/2}	-3.95513593304
3	2, -3	2d _{5/2}	-3.94046656480	1, 4	1g _{7/2}	-3.94046656480	2, -3	2d _{5/2}	-3.9421162245	1, 4	1g _{7/2}	-3.93838260791
4	2, -4	2f _{7/2}	-3.92071895908	1, 5	1h _{9/2}	-3.92071895908	2, -4	2f _{7/2}	-3.92280064576	1, 5	1h _{9/2}	-3.91822686782
$C_{\text{ps}} = -5, r_e = 0.4$												
1	1, -1	1s _{1/2}	-3.98445312686	0, 2	0d _{3/2}	-3.98445312686	1, -1	1s _{1/2}	-3.98511403062	0, 2	0d _{3/2}	-3.98329452394
2	1, -2	1p _{3/2}	-3.97358257158	0, 3	0f _{5/2}	-3.97358257158	1, -2	1p _{3/2}	-3.97473339040	0, 3	0f _{5/2}	-3.97198877105
3	1, -3	1d _{5/2}	-3.9592728873	0, 4	0g _{7/2}	-3.9592728873	1, -3	1d _{5/2}	-3.96087984589	0, 4	0g _{7/2}	-3.95725033962
4	1, -4	1f _{7/2}	-3.94164843833	0, 5	0h _{9/2}	-3.94164843833	1, -4	1f _{7/2}	-3.94369711585	0, 5	0h _{9/2}	-3.93920334486
1	2, -1	2s _{1/2}	-3.9687414894	1, 2	1d _{3/2}	-3.9687414894	2, -1	2s _{1/2}	-3.96945941660	1, 2	1d _{3/2}	-3.96747211076
2	2, -2	2p _{3/2}	-3.95599216066	1, 3	1f _{5/2}	-3.95599216066	2, -2	2p _{3/2}	-3.95724655890	1, 3	1f _{5/2}	-3.95424678741
3	2, -3	2d _{5/2}	-3.93934209634	1, 4	1g _{7/2}	-3.93934209634	2, -3	2d _{5/2}	-3.94107637432	1, 4	1g _{7/2}	-3.93715457112
4	2, -4	2f _{7/2}	-3.91920194081	1, 5	1h _{9/2}	-3.91920194081	2, -4	2f _{7/2}	-3.92138852243	1, 5	1h _{9/2}	-3.91658741001

4. Conclusion

We have obtained the approximate energy equation and the corresponding wave functions of the Dirac equation for the shifted Deng–Fan potential coupled with the Yukawa-like tensor under the conditions of the spin and pseudospin symmetry. After using an improved approximation scheme, the resulting Schrödinger-like equation is solved by using the elegant NU method. Our numerical data describe the energy splitting in detail. In Table 1, in the case of the spin symmetry, for $r_e = 0.4 \text{ fm}^{-1}$ and in the absence of tensor interaction, i.e., $A = 0$ (when the tensor potential vanishes), we noticed the degeneracy between two states in the spin symmetry doublets as follows:

$$n \begin{Bmatrix} p_{3/2} \\ d_{5/2} \\ f_{7/2} \\ g_{9/2} \end{Bmatrix} = n \begin{Bmatrix} p_{1/2} \\ d_{3/2} \\ f_{5/2} \\ g_{7/2} \end{Bmatrix}.$$

Also, the system becomes less attractive when the quantum numbers increase, i.e., the energy levels are negative. By reducing equilibrium inter-nuclear distance by half yields a very small increment in the energies. It is also noticed that the presence of tensor interaction, say $A = 0.5$, removes the degeneracy between the states. When $C_s = 0$, we noticed that the energy levels are repulsive and becoming weakly repulsive with quantum numbers increasing.

Furthermore, in Table 2, we take $C_{ps} = 0$ (pseudospin symmetry) and $C_{ps} = -5$. For $C_s = 0$, $r_e = 0.8 \text{ fm}^{-1}$ and $A = 0$, we noticed that the system is repulsive and also the degeneracy between two states in the pseudospin symmetry doublets can be written as follows:

$$n \begin{Bmatrix} s_{1/2} \\ p_{3/2} \\ d_{5/2} \\ f_{7/2} \end{Bmatrix} = (n-1) \begin{Bmatrix} d_{3/2} \\ f_{5/2} \\ g_{7/2} \\ h_{9/2} \end{Bmatrix}.$$

The same behavior is also observed by reducing the equilibrium inter-nuclear distance by half of the initial. Also, similar to the spin symmetry case, the decrement in r_e results in very small increment in the relativistic energy spectrum. Again, we obtain energy spectrum in the presence of the Yukawa tensor term. As it can be observed from the results presented in Table 2, the degeneracy between the states are removed in this regard. In addition, the system becomes less attractive when the quantum numbers increase.

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Appendix A. The NU Method

One of the computational tools utilized in solving the Schrödinger-like equation including the centrifugal barrier and/or the spin-orbit coupling term is called the NU method. Proposed in order to solve the second-order differential wave equation of the hypergeometric-type:

$$\Psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\Psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\Psi(z) = 0, \quad (\text{A.1})$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most second degree, and $\tilde{\tau}(z)$ is a polynomial of first degree.^{33,34,49,61–63,77,83,84,86–92}

The prime denotes the differentiation with respect to z . In finding a particular solution to (A.1), one needs to decompose the wave function $\psi(z)$ as

$$\psi(z) = \phi(z)y(z), \quad (\text{A.2})$$

yielding the following hypergeometric type equation:

$$\sigma(z)y''(z) + \tau(z)y'(z) + \Lambda y(z) = 0, \quad (\text{A.3})$$

where

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z) \quad (\text{A.4})$$

satisfies the condition $\tau'(z) < 0$, which will have a negative derivative and $\pi(z)$ is related to the function $\phi(z)$ by

$$\pi(z) = \sigma(z) \frac{d}{dz} [\ln \phi(z)]. \quad (\text{A.5})$$

The function π and the parameter λ required for this method are defined as follows:

$$\pi(z) = \left(\frac{\sigma' - \tilde{\sigma}}{2} \right) \pm \sqrt{\left(\frac{\sigma' - \tilde{\sigma}}{2} \right)^2 - \tilde{\sigma} + k\sigma}. \quad (\text{A.6})$$

$$\lambda = k + \pi'(z). \quad (\text{A.7})$$

In order to find the value of k , the expression under the square root must be a square of a polynomial. This gives the polynomial $\pi(z)$ which is dependent on the transformation function $z(r)$. Also the parameter λ defined in Eq. (A.7) takes the form:

$$\lambda_n = -n\tau' - \left[\frac{n(n-1)}{2} \sigma'' \right], \quad n = 0, 1, 2, \dots \quad (\text{A.8})$$

The polynomial solutions $y_n(z)$ are given by the Rodrigue relation:

$$y_n(z) = \frac{C_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z) \rho(z)], \quad (\text{A.9})$$

where C_n is a normalization constant and the weight function $\rho(z)$ satisfies the following relation:

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z). \quad (\text{A.10})$$

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