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Condensed Kannan-Type Maps and Their Efficiency Measures in Complete Metric Spaces

Olalekan Taofeek Wahab*
Department of Mathematics and Statistics, Kwara State University, Malete, Nigeria
e-mail: taofeek.wahab@kwasu.edu.ng

Garba Risqot Ibrahim
Department of Mathematics and Statistics, Kwara State University, Malete, Nigeria
e-mail: risqot.ibrahim@kwasu.edu.ng

Salaudeen Alaro Musa
Department of Mathematics and Statistics, Kwara State University, Malete, Nigeria
e-mail: salaudeen.musa@kwasu.edu.ng

Abstract. Recently, some studies introduced interpolative Kannan-type maps for solving non-unique fixed-point problems. However, these studies do not offer a technique for determining the fractional powers of the maps, which would be useful in approximating the fixed points. This study uses two symmetry terms of the conventional Kannan contraction to define a novel condensed Kannan-type contraction in complete metric spaces. By imposing the condensed map on the Picard operator, we prove the existence of unique and non-unique fixed points. We also define a criterion for selecting an appropriate real constant α suitable for approximating the fixed points of the condensed map. We consider two practical and numerical examples to show both the versatility and validity of the hypotheses of this study. These results show that the class of condensed maps is strictly larger than the existing class Kannan-type maps, ensure suitability for solving both unique and non-unique fixed points, and do so with a better convergence rate.

1. Introduction

Let M be a nonempty set, $k : M \times M \to \mathbb{R}^+$ be a metric function, and (M, k) be a complete metric space. In [2], Banach proved a fundamental result in the area of fixed point theory in (M, k) by introducing a self-map $T : (M, k) \to (M, k)$ that satisfies both contraction and continuity properties. Contrary to the result of Banach, Kannan

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[4] introduced another significant result that ensures the existence of fixed points of a self-map that is both contractive and discontinuous in (M, k). Kannan's postulation was one of the first classes of nonlinear contractive mappings, as described in [11]. Several possibilities for understanding the approximation and existence of fixed points of discontinuous maps have emerged since the introduction of Kannan's result.

Theorem 1.1. [4] *Let (M, k) be a complete metric space and $T : M \to M$ be a Kannan contraction mapping,*

$$k(Tu, Tv) \leq \lambda \left[k(u, Tu) + k(v, Tv)\right], \ \forall u, v \in M; \lambda \in [0, \frac{1}{2}). \tag{1}$$

Then, T has a unique fixed point.

The above result has been extended in many ways in the past decades, as is evidenced by its over 2160 citations on Google Scholar. There has been a signicant increase in citations since 2010, and this is due largely to recent efforts to improve condition (1). Karapinar [5] defined a Kannan-type map, called the interpolative Kannan-type contraction, which depends on the a real contracting constant and fractional powers as follows:

$$k(Tu, Tv) \leq \mu k(u, Tu)^{\alpha} k(v, Tv)^{1-\alpha}, \ \forall u, v \in M; \tag{2}$$

for $\mu \in [0, 1)$ and $\alpha \in (0, 1)$. In [6], Karapinar et al. pointed out that condition (2) is only suitable to solve non-unique fixed point problems. Since then, several authors have generalized the interpolative Kannan-type map in various distance spaces (see [1, 3, 8, 10, 9]). These results largely depends on fractional powers, but there is no technique for selecting a suitable power to approximate the fixed points of operators under the interpolative maps. Therefore, there is a need to answer the following questions (Qs):

- Q_1 If condition (2) is satisfied for an $\alpha \in (0, 1)$, does it form an interpolative Kannan-type map for every $\alpha \in (0, 1)$?
- Q_2 At what interval(s) of α does an operator form an interpolative Kannan-type map?
- Q_3 Is there any genuine comparison of (2) for each $\alpha \in (0, 1)$ that also maximizes convergent rate?

While we swiftly answer the first question in the next two examples, the main goal in this study is to answer the three Qs in the affirmative.

Example 1.2. [5] Let $M = \{x, y, z, w\}$ be endowed with a metric k such that $k(x, x) = k(y, y) = k(z, z) = k(w, w) = 0$, $k(x, y) = 3$, $k(x, z) = 4$, $k(x, w) = \frac{5}{2}$, $k(y, z) = \frac{3}{2}$, $k(y, w) = 2$, and $k(z, w) = \frac{3}{2}$. Define T on M by $T : \begin{pmatrix} x & y & z & w \\ x & w & x & y \end{pmatrix}$. Then, does T form an interpolative Kannan-type for all $\alpha \in (0, 1)$?

The answer is false. To see this, we consider the six possibilities of map T on M. Of all these possibilities, the cases $\{x, y\}$, $\{x, w\}$, $\{x, z\}$, $\{w, y\} \subset M$ fail to satisfy (2). However, for the case $\{z, w\} \subset M$, we have $k(Tw, Tz) \leq \mu k(w, Tw)^{\alpha} k(z, Tz)^{1-\alpha}$.

Thus, T forms the interpolative Kannan-type for $\alpha \in (0, 0.33)$ and fails to satisfy (2) for $\alpha \in (0.33, 1)$.
Also for $\{y, z\} \subset M$, T does not form the interpolative Kannan-type for $\alpha \in (0.66, 1)$.

Example 1.3. Let $M = [-1, 1]$ be given by the usual metric and let $T : M \to M$ is defined by

$$Tu = \begin{cases} \frac{u}{3} \text{ if } u \in \{-1, 1\}; \\ \\ \frac{1}{3} \text{ if } u \in (-1, 1). \end{cases}$$

Then, T does not form the interpolative Kannan-type map since there is no α and μ for which (2) is satisfied.
Indeed, there were attempts to answer the Q_3 by previous authors, for example, Mitrovic et al. [7] introduced a hybrid weight-type contraction in b-metric space, comprising of Reich-Rus-Ciric and Reich-Rus-Ciric by interpolation as follows:

$$k(Tu, Tv) \leq \mu \mathcal{M}^r(T, u, v, a) \tag{3}$$

where

$$\mathcal{M}^r(T, u, v, a) = \begin{cases} [a_1(k(u, v))^r + a_2(k(u, Tu))^r + a_3(k(v, Tv))^r]^{\frac{1}{r}}, r > 0; \\ \\ (k(u, v))^{a_1}(k(u, Tu))^{a_2}(k(v, Tv))^{a_3}, \ r = 0; \end{cases}$$

for all $u, v \in M \backslash Fix(T)$, $0 \leq a = (a_1, a_2, a_3)$ with $a_1 + a_2 + a_3 = 1$. Their result (and its numerous generalizations in the recent literature) generalized the linear, quasilinear, and quadratic Kannan-type but neither did they compare nor measure the discrepancies between them, thus leaving the above Q_3 unanswered to date. Therefore, this paper aims to (i) present a more robust hybrid Kannan-type map that improves and compares favorably with the existing Kannan-type maps; (ii) suggest a technique for examining the existence properties of unique and non-unique fixed point problems; and (iii) provide a criterion to determine the suitable powers for approximating the fixed points.

2. Our Main Result

In this section, we first give a definition that is a useful variant for studying the results of this paper. We define a metrical term $k(Tu, Tv)$ as two Kannan-type conditions. In practice, if the metrical term lies between the two conditions, then it will allow us to conclude that the map T exists.

Definition 2.1. Let (M, k) be a metric space. A map $T : M \to M$ will be called a condensed Kannan-type contraction if there exist constants $\alpha \in (0, 1)$, $\lambda \in [0, \frac{1}{2})$, and $\mu \in [0, 1)$, such that

$$k(Tu, Tv) \begin{cases} \leq \lambda \left[k(u, Tu)^{2\alpha} + k(v, Tv)^{2(1-\alpha)}\right], & \text{(CK1)} \\ \\ \geq \mu k(u, Tu)^{\alpha} k(v, Tv)^{1-\alpha}, & \text{(CK2)} \end{cases} \tag{4}$$

for all $u, v \in M$ (or $M \backslash Fix(T)$), and contrariwise.

The condition (4) is valid if the fractional powers α and $1-\alpha$ are interchanged. The following lemma with obvious proof is also of importance.

Lemma 2.2. *A sequence $\{u_n\}$ in (M,k) will be called a Cauchy sequence if for each $j,k \in \mathbb{N}_0$ with $k > j$, we have*

$$\lim_{n\to\infty} k(u_{n+j}, u_{n+k}) = 0 \text{ for all } n \in \mathbb{N}.$$

The existence theorem of an operator satisfying (4) is presented as follow:

Theorem 2.3. *Let (M,k) be a complete metric space and $T : M \to M$ be a condensed Kannan-type contraction map. Then, T has a unique fixed point.*

Proof. Let $u_0 \in M$ and define the sequence $\{u_n\} \subset M$ by the Picard operator $u_{n+1} = Tu_n$ for all $n \in \mathbb{N}_0$. If there exists $n_0 \in \mathbb{N}_0$ such that $u_{n_0+1} = Tu_{n_0} = u_{n_0}$, then the proof is immediate. Now, let $u_{n+1} \neq u_n$ for each $n \in \mathbb{N}_0$; using the (CK1) of (4), we get

$$\begin{aligned} k(u_n, u_{n+1}) &= k\left(Tu_{n-1}, Tu_n\right) \\ &\leq \lambda\left[k(u_{n-1}, Tu_{n-1})^{2\alpha} + k(u_n, Tu_n)^{2(1-\alpha)}\right] \\ &= \lambda\left[k(u_{n-1}, u_n)^{2\alpha} + k(u_n, u_{n+1})^{2(1-\alpha)}\right] \end{aligned} \tag{5}$$

By using the (CK2) and (5), this gives

$$\begin{aligned} 0 &\leq k(u_n, u_{n+1}) - \mu k(u_{n-1}, u_n)^{\alpha} k(u_n, u_{n+1})^{1-\alpha} \\ &\leq \lambda\left[k(u_{n-1}, u_n)^{\alpha} - k(u_n, u_{n+1})^{1-\alpha}\right]^2 \text{ with } \mu \equiv 2\lambda. \end{aligned} \tag{6}$$

Furthermore, we have

$$\left[k(u_{n-1}, u_n)^{\alpha} - k(u_n, u_{n+1})^{1-\alpha}\right]^2 \geq 0 \tag{7}$$

Two cases are recorded from the inequality (7), namely:

(a) $k(u_{n-1}, u_n)^{\alpha} \leq k(u_n, u_{n+1})^{1-\alpha}$; and
(b) $k(u_{n-1}, u_n)^{\alpha} \geq k(u_n, u_{n+1})^{1-\alpha}$.

Case (a): If $k(u_{n-1}, u_n)^{\alpha} \leq k(u_n, u_{n+1})^{1-\alpha}$, then inequality (6) becomes

$$k(u_n, u_{n+1})^{2\alpha-1} \leq \mu < 1,$$

which is a contradiction for $\alpha > 0$. Thus, we cannot proceed with case (a).
Case (b): If $k(u_{n-1}, u_n)^{\alpha} \geq k(u_n, u_{n+1})^{1-\alpha}$, then inequality (6) gives

$$k(u_n, u_{n+1}) \leq \mu k(u_{n-1}, u_n)^{2\alpha}$$

Inductively, we have

$$k(u_n, u_{n+1}) \leq \mu^{\sum_{i=0}^{n-1}(2\alpha)^i} k(u_0, u_1)^{(2\alpha)^n}. \tag{8}$$

If $\alpha < \frac{1}{2}$, then we can conclude the proof. Now, let $\alpha \geq \frac{1}{2}$ and taking limit as $n \to \infty$ over the last inequality, we have that

$$\lim_{n\to\infty} k(u_n, u_{n+1}) \leq \lim_{n\to\infty} \mu^{\sum_{i=0}^{n-1}(2\alpha)^i} k(u_0, u_1)^{(2\alpha)^n}.$$

Since $\mu < 1$, $k(u_0, u_1)$ is fixed, and $\sum_{i=0}^{n-1}(2\alpha)^i = \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} k(u_n, u_{n+1}) = 0.$$

Moreover, for each $j, k \in \mathbb{N}_0$ such that $k > j$, we have

$$\lim_{n\to\infty} k(u_{n+j}, u_{n+k}) = 0.$$

By Lemma 2.2, this proves that $\{u_n\}$ is a Cauchy sequence. By the completeness of (M, k), u_n converges to a point u^*(say) in M.
That is,

$$\lim_{n\to\infty} u_n = u^*.$$

To show that the point u^* coincides with the map T is to deduce that $k(Tu^*, u^*) = 0 \Leftrightarrow u^* = Tu^*$. Indeed,

$$k(Tu^*, u^*) \leq k(Tu^*, Tu_n) + k(u_{n+1}, u^*) \tag{9}$$

and

$$\begin{aligned} k(Tu^*, Tu_n) &\leq \lambda \left[k(u^*, Tu^*)^{2\alpha} + k(u_n, Tu_n)^{2(1-\alpha)}\right] \\ &= \lambda \left[k(u^*, Tu^*)^{2\alpha} + k(u_n, u_{n+1})^{2(1-\alpha)}\right] \end{aligned}$$

By taking limit as $n \to \infty$ across (9), this gives the desired result.
Suppose that v^* is any other fixed point of T such that $k(u^*, v^*) \neq 0 \Leftrightarrow u^* \neq v^*$. Then,

$$\begin{aligned} k(u^*, v^*) = k(Tu^*, Tv^*) &\leq \lambda[k(u^*, Tu^*)^{2\alpha} + k(v^*, Tv^*)^{2(1-\alpha)}] \\ &= \lambda[k(u^*, u^*)^{2\alpha} + k(v^*, v^*)^{2(1-\alpha)}] = 0. \end{aligned}$$

This leads to a contradiction. Hence, $u^*(= v^*)$ is unique. □

Remark 2.4. Theorem 2.3 unifies both the Kannan contraction (1) and the interpolative Kannan-type (2) as presented in the following Corollary.

Corollary 2.5. *Let (M, k) be a complete metric space and $T : M \to M$ be a condensed Kannan-type contraction map, for all $u, v \in M$ (or $M \backslash Fix(T)$),*

$$k(Tu, Tv) \begin{cases} \leq \lambda \left[k(u, Tu) + k(v, Tv)\right], \\ \\ \geq \mu k(u, Tu)^{\frac{1}{2}} k(v, Tv)^{\frac{1}{2}} \end{cases} \tag{10}$$

and contrariwise, where $\lambda \in [0, \frac{1}{2})$ and $\mu \in [0, 1)$. Then, T has a unique fixed point.

Proof. Obviously, the proof is immediate from Theorem 2.3 if $\alpha = \frac{1}{2}$. □

The following corollary also holds if we drop the interpolative condition in (4).

Corollary 2.6. *Let (M,k) be a complete metric space and $T : M \to M$ is a condensed Kannan-type (from right) contraction map*

$$k(Tu,Tv) \leq \lambda \left[k(u,Tu)^{2\alpha} + k(v,Tv)^{2(1-\alpha)}\right], \tag{11}$$

where $\lambda \in [0, \frac{1}{2})$, $\alpha \in (0,1)$ and for all $u, v \in M$. Then, T has a unique fixed point.

Remark 2.7. If for some $\alpha \in (0,1)$ a self-operator satisfies (2), then inequality (11) (in Corollary 2.6) is a refinement of condition (2). This is presented in the next theorem.

Theorem 2.8. *Let (M,k) be a complete metric space and $T : M \to M$ is a refined Kannan-type contractive map*

$$k(Tu,Tv) \leq \mu k(u,Tu)^{\alpha} k(v,Tv)^{1-\alpha} \leq \lambda \left[k(u,Tu)^{2\alpha} + k(v,Tv)^{2(1-\alpha)}\right] \tag{12}$$

where $\mu \in (0,1)$, $\lambda \in (0, \frac{1}{2})$, $\alpha \in (0,1)$ and for all $u, v \in M$ (or $M \backslash Fix(T)$). Then, T has a unique fixed point.

Proof. The proof of first inequality in (12) follows from the result in [5], while the second inequality is immediate from the Case (b) above. □

Remark 2.9. Theorem 2.3, Corollaries 2.5 & 2.6 are suitable for studying the self-operator that does not satisfy some previous Kannan-type maps.

We consider the following unique fixed point problem to justify the Remark 2.9.

Example 2.10. Let $M = [0,1]$ be endowed with the metric $k(u,v) = |x - y|$ and let $T : [0,1] \to [0,1]$ be a self-map given by

$$Tu = \begin{cases} \frac{u}{3} \text{ if } u \in \{0,1\}; \\ \\ \frac{1}{3} \text{ if } u \in (0,1). \end{cases}$$

with $Fix(T) = \{0, \frac{1}{3}\}$. Then, the map T

i. does not satisfy the Kannan contraction (1).
ii. does not form the interpolative Kannan-type (2).
iii. does not satisfy the hybrid weight type (3).
iv. satisfies the condensed Kannan-type.

Firstly, if $u = v$ or $u, v \in (0,1)$, all of the above-mentioned conditions are satisfied. Now, suppose that $u \neq v$ for $u, v \in \{0,1\}$, then the left hand side inequality (LHSI) for these conditions is $\frac{1}{3}$. By letting $\alpha = \frac{4}{5}$, $\lambda = \frac{5}{11}$, and $\mu = \frac{10}{11}$, we compare the LHSI with the right hand side inequalities (RHSIs) (1), (2), (3), (4) to know the dominant condition.

In Table 1, we carefully choose all parameters of these inequalities so that they compare side by side with one another. For instance, we let $r = \frac{2}{5}$ in (3) with $a = (0, \frac{1}{2}, \frac{1}{2})$ (see Definition 2.1, [7]) to have the same fractional powers like that of the condensed map. Of all the Kannan-type maps considered, only the condensed Kannan-type is suitable for studying the operator T in Example 2.10. In addition,

TABLE 1. Comparison of various Kannan-type maps for Example 2.10

S/R	CONDITIONS	LHSI	RHSIs	VALIDITY
i.	Kannan contraction (1)	0.333333	0.303030	No
ii.	Interpolative Kannan (2)	"	0.000000	No
iii.	Hybrid weight type (3) ($r = \frac{2}{5}$)	"	0.107137	No
iv.	condensed Kannan-type	"	0.386492	Yes

if we let $r = 2$ in (3) with $a = (0, \frac{1}{2}, \frac{1}{2})$, there are some $\alpha \in (0, 1)$ for which the condensed Kannan-type dominates (3).
We equally consider a non-unique fixed point problem as follows:

Example 2.11. Let M be given as contained in Example 1.3 and $T : M \to M$ is given by

$$Tu = \begin{cases} \frac{u}{3} \text{ if } u \in \{-1, 1\}; \\ \\ \frac{1}{3} \text{ if } u \in (-1, 1). \end{cases}$$

Then, the map T

i. does not satisfy the Kannan contraction (1).
ii. does not form the interpolative Kannan (2).
iii. does not satisfy the hybrid weight type (3).
iv. satisfies the condensed Kannan-type.

To see this, we let $\alpha = \frac{1}{10}$, $\lambda = \frac{49}{100}$ and $\mu = \frac{19}{20}$. As presented in Table 2, only

TABLE 2. Comparison of various Kannan-type maps for Example 2.10

S/R	CONDITIONS	LHSI	RHSIs	VALIDITY
i.	Kannan contraction (1)	0.666666	0.633333	No
ii.	Interpolative Kannan (2)	"	0.633333	No
iii.	Hybrid weight type (3) for all $r > 0$	"	0.633333	No
iv.	condensed Kannan-type	"	0.697319	Yes

the condensed Kannan-type is suitable for studying the operator T in Example 2.11. More so, T fails to satisfy the hybrid weight type (3) with $a = (0, \frac{1}{2}, \frac{1}{2})$, for all $r > 0$.

Remark 2.12. In the results presented by Table 1 and 2, we observed, respectively, that

i. the LHSI lies between the condensed Kannan-type conditions, that is, $0.333333 \in (0, 0.386492)$ for Example 2.10.
ii. the LHSI $0.666666 \in (0.633333, 0.697319)$ for Example 2.11.

In riddance to Remark 2.12, the next obvious questions are that: (i) At what value(s) of α would the tripartite of the condensed Kannan-type map meet? (ii) At what interval(s) of α does an operator satisfy the condensed Kannan-type map? The affirmative answers are in the next section.

3. Efficiency Measures of the Condensed Kannan-Type Map

Here we study yet another variant of the condensed Kannan-type to determine the closeness of the measured conditions in (4) to an actual value. This will enable one to examine the stretch or compress effects relative to the constants α, which in turn, contribute to the optimization of convergent rate. Also, this technique will suggest an approach for obtaining suitable values of $\alpha \in (0,1)$ that are reliable for approximating nonlinear operators under the condensed map. Ill-suited accuracy leads to non-reliability and while a moderate accuracy implies reliability of the condensed map. In what follows, we define a measure for the condensed Kannan-type map with a metrical notion.

Definition 3.1. Let $\Gamma : \mathbb{R}^+ \times \mathbb{R}^+ \times (0,1) \to \mathbb{R}^+$ be a multivalued function. The α-measure of the condensed Kannan-type map, denoted by σ_α, will be defined as a non-negative real number

$$\sigma_\alpha = |\Gamma(s,t,\alpha)|,$$

where $\Gamma(s,t,\alpha) = s^\alpha - t^{1-\alpha}$, for all $s,t \in \mathbb{R}^+$ with $s \neq t$ and $\alpha \in (0,1)$.

Here both s and t are, in fact, the distance function $k : M \times T(M) \to \mathbb{R}^+$, where $T(M)$ is a subspace of M. Immediate consequences of Definition 3.1 are as follows:

i. If $\sigma_\alpha = 0$, then the condensed Kannan-type map is highly reliable. In this case, the tripartite of the condensed condition meet at the same point.
ii. If $0 < \sigma_\alpha < 1$, the condensed Kannan-type map is reliable.
iii. If $\sigma_\alpha \geq 1$, the condensed Kannan-type map is not reliable.

Theorem 3.2. *Let (M,k) be a complete metric space and $T(M)$ be a subspace of M. The map T is a condensed Kannan-type map if and only if there exists an $\alpha \in (0,1)$ such that $\sigma_\alpha = 0$.*

Proof. Assume that there exists an $\alpha \in (0,1)$ such that $\sigma_\alpha = 0$, by letting $s = k(u,Tu)$ and $t = k(v,Tv)$ for $u,v \in M$, we have $k(u,Tu)^\alpha = k(v,Tv)^{1-\alpha}$. It suffices to show that for $\{u_n\} \subset \mathcal{M}$, $k(u_n,u_{n+1}) = 0$ as $n \to \infty$. For $\{u_n\} \subset \mathcal{M}$ such that $u_n \neq u_{n+1} = Tu_n$ and letting $u = u_{n-1}$, $v = u_n$. From both (CK1) and (CK2), we have

$$\mu k(u_{n-1},Tu_{n-1}) \leq k(Tu_{n-1},Tu_n) \leq \mu k(u_{n-1},Tu_{n-1})^{2\alpha}$$

or equivalently

$$\mu k(u_{n-1},u_n) \leq k(u_n,u_{n+1}) \leq \mu k(u_{n-1},u_n)^{2\alpha}$$

Taking limit as $n \to \infty$ across, this gives the desire result.
On other hand, let T be a condensed Kannan-type map. Let $u_n \in \mathcal{M}$ such that $u_n \neq u_{n+1}$ for $n \in \mathbb{N}$, we have from (CK1) and (CK2) that

$$k(u_n, u_{n+1}) \leq \lambda \left[k(u_{n-1}, u_n)^{2\alpha} + k(u_n, u_{n+1})^{2(1-\alpha)} \right] \tag{13}$$

and

$$k(u_n, u_{n+1}) \geq \mu k(u_{n-1}, u_n)^{\alpha} k(u_n, u_{n+1})^{1-\alpha} \tag{14}$$

respectively. By letting $s_n = k(u_{n-1}, u_n)$, $t_n = k(u_n, u_{n+1})$, and combining both (13) and (14), this gives

$$\sigma_\alpha^2 \equiv \left(s_n^{\alpha} - t_n^{1-\alpha} \right)^2 \geq 0, \text{ where } \sigma_\alpha \equiv s_n^{\alpha} - t_n^{1-\alpha}. \tag{15}$$

We have two cases arise from inequality (15), that is, $s_n^{\alpha} \leq t_n^{1-\alpha}$ and $s_n^{\alpha} \geq t_n^{1-\alpha}$.
By inserting $s_n^{\alpha} \leq t_n^{1-\alpha}$ into (14), we have

$$t_n \geq \mu t_n^{2(1-\alpha)} \tag{16}$$

Solving (13) and (16), we get a recursive relation

$$t_n^{\alpha} \leq \left(\frac{\lambda}{\mu - \lambda} \right)^{\frac{1}{2}} s_n^{1-\alpha} \tag{17}$$

Similarly, by inserting $s_n^{\alpha} \geq t_n^{1-\alpha}$ into (14), and then solve using (16), we get

$$s_n^{1-\alpha} \leq \left(\frac{\lambda}{\mu - \lambda} \right)^{\frac{1}{2}} t_n^{\alpha} \tag{18}$$

By subtracting (18) from (17), we obtain

$$(t_n^{\alpha} - s_n^{1-\alpha})^2 \leq \frac{\lambda}{\mu - \lambda} (s_n^{1-\alpha} - t_n^{\alpha})^2$$

or

$$\sigma_\alpha^2 \leq \frac{\lambda}{\mu - \lambda} \sigma_\alpha^2 \tag{19}$$

Since $\lambda \in (0, \frac{1}{2})$ and $\mu \in (0, 1)$, then $\frac{\lambda}{\mu - \lambda} < 1$. Thus,

$$\sigma_\alpha \leq \left(\frac{\lambda}{\mu - \lambda} \right)^{\frac{1}{2}} \sigma_\alpha < \sigma_\alpha$$

leads to a contradiction. Hence, $\sigma_\alpha = 0$. □

Remark 3.3. We note here that if there exists $\alpha^* \in (0, 1)$ such that $\sigma_{\alpha^*} = 0$, the coordinate $(\alpha^*, \sigma_{\alpha^*})$ is a stationary value.

As application of Theorem 3.2, we present two numerical examples (extract of Examples 1.2, 1.3) to demonstrate the reliability of operators under the condensed Kannan-type conditions.

Example 3.4. Let T be defined on M as presented in Example 1.2. Let $\{y, z\} \subset M$, then

$$k(Ty, Tz) \begin{cases} \leq \lambda[k(v, Tv)^{2\alpha} + k(z, Tz)^{2(1-\alpha)}] \\ \geq \mu k(v, Tv)^{\alpha} k(z, Tz)^{1-\alpha}. \end{cases}$$

or otherwise.

By interchanging the powers of the last inequality, we demonstrate the behavior of σ_α against α and $1-\alpha$. In Figure 1 (a), the map T is reliable for $\alpha \in (0, 0.66)$ and

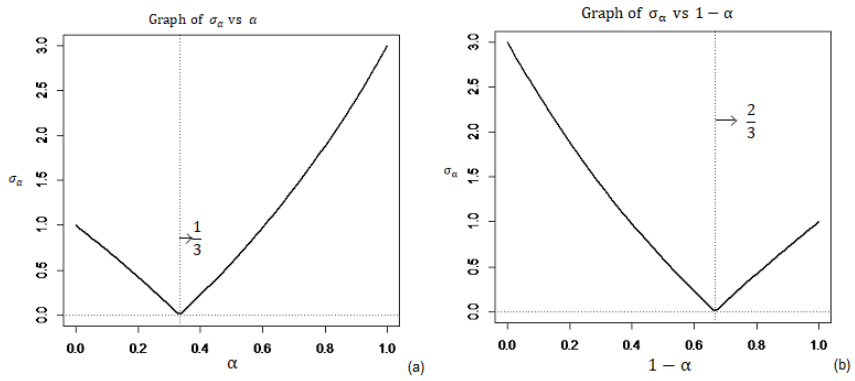


FIGURE 1. Graph of σ_α vs α and $1-\alpha$.

not reliable for $\alpha \in (0.66, 1)$. Also, σ_α attains its minimum value at $(\alpha^*, \sigma_\alpha^*) = (\frac{1}{3}, 0)$. Thus, T is highly reliable at $\alpha^* = \frac{1}{3}$. In Figure 1 (b), σ_α is asymmetrically reversed as T is not reliable when $\alpha \in (0, 0.33)$ and reliable for $\alpha \in (0.33, 1)$. In this case, σ_α attains its minimum at $(\alpha^*, \sigma_\alpha^*) = (\frac{2}{3}, 0)$ wherein T is highly reliable. Figure 2

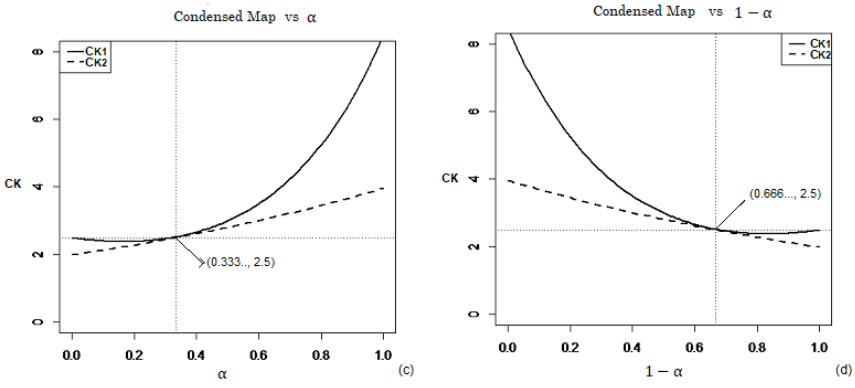


FIGURE 2. Graph of condensed Kannan-type map vs α and $1-\alpha$.

(c) and (d) show that the tripartite terms of the condensed map meet at 2.5 for the stationary values $(\frac{1}{3}, 0)$ and $(\frac{2}{3}, 0)$, respectively.

Example 3.5. Let $T : M \to M$ be a self-map as presented in Example 2.11:

$$Tu = \begin{cases} \frac{u}{3} \text{ if } u \in \{-1, 1\}; \\ \\ \frac{1}{3} \text{ if } u \in (-1, 1). \end{cases}.$$

For $u, v \in \{-1, 1\}$ with $u \neq v$, we visualize the behavior of σ_α against α. As shown

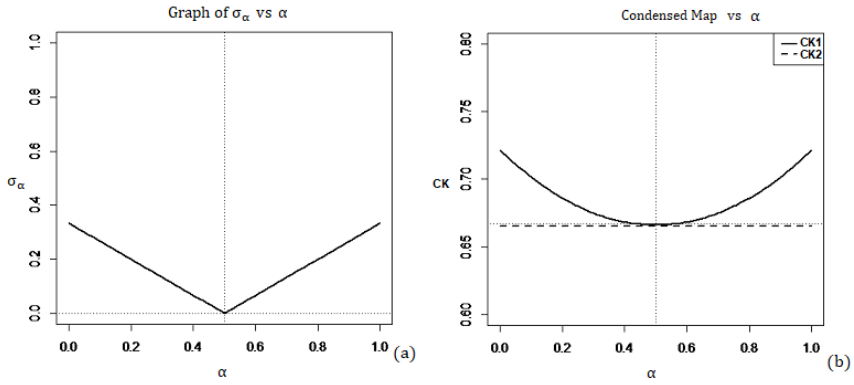


FIGURE 3. (a) Graph of σ_α vs α. (b) Graph of condensed Kannan-type map vs α.

in Figure 3 (a), the map T satisfies the condensed Kannan-type for all $\alpha \in (0, 1)$. This is so because σ_α lies in the interval $(0, 1)$ and attains its minimum value at $(\alpha^*, \sigma^*_\alpha) = (\frac{1}{2}, 0)$.
Also in Figure 3 (b), the value of the tripartite terms of (4) (at $\alpha = \frac{1}{2}$) is $\frac{2}{3}$. Finally in Figure 4, we plot a graph of estimate (8) against $\mu \in (0, 1)$ to visualize the performance of (4) for Example 3.4. The performance gradually becomes better as $\alpha \to \frac{1}{2}^-$.

4. Conclusion

This paper introduced a new condensed Kannan-type map for studying the existence properties of operators with unique and non-unique fixed points. We study the reliability of the condensed map concerning the fractional power α using a technique called the $\alpha-$measure. This technique suggests all suitable values of α that are appropriate for approximating fixed points of operators and also measures the discrepancies among various Kannan-type maps. We considered two fixed points problems (Examples 2.10 and 2.11) to validate the results. Both examples fulfilled all the hypotheses of Theorem 2.3 and Corollary 2.5 while other previous Kannan-type maps do not. This indicated that the condensed type map is strictly a larger class than some of

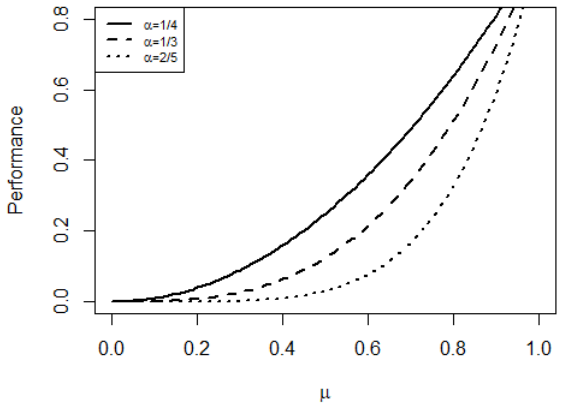


FIGURE 4. Performance estimate vs μ.

the existing Kannan-type maps. Also, the performance of the condensed-type map becomes better when α approaches the critical value from the left. Using this paper's technique, further studies could examine the discrepancies between various recent Kannan-type maps in the literature.

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