
EXACT SOLUTIONS OF THE DUFFIN-KEMMER-PETIAU EQUATION FOR THE SCARF-GROSCHÉ POTENTIALS[†]

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Abstract

The solutions of the Duffin-Kemmer-Petiau equation are obtained for a set of the Scarf-Grosché potentials. The energy eigenvalues and wave functions for the Scarf-Grosché potentials are obtained for equal scalar and vector potentials using the improved asymptotic iteration method. The results of the two out of these potentials are deduced from other two potentials, respectively. The numerical results of the energy eigenvalues are also obtained.

Keywords: Duffin-Kemmer-Petiau Equation, Scarf-Grosché potentials, Asymptotic iteration method, wave functions, eigenvalues, Jacobi polynomials.

PACs: 03.65.Pm, 03.65.Ca, 03.65.Ge

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1. INTRODUCTION

There has been, in recent years, a renewed interest in the study of the Duffin-Kemmer-Petiau equation (Okninski, 2004). The Duffin-Kemmer-Petiau equation is used to describe spin-0 and spin-1 particles just as the Dirac equation is used for describing spin-1/2 particles (Yahya and Oyewumi, 2013). Exact solutions of the Duffin-Kemmer-Petiau equation in the presence of an external field have been investigated by some authors (Nedjadi and Barrett, 1993; Nedjadi and Barrett, 1998; Sogut et al, 2002; Yasuk et al, 2008). The Duffin-Kemmer-Petiau equation has been applied to QCD (large and short distances) by Gribov (1999), to covariant Hamiltonian dynamics (Kanatchikov, 2000), and has been generalized to curve space-time (Hassanabadi et al, 2012; Lunardi et al, 2001). This equation for spin-0 bosons under a vector potential possesses the same mathematical structure of its well known partner, the Klein-Gordon equation (Yazarloo et al, 2013).

The Scarf-Grosche potentials were introduced originally by Grosche (1993) and are given as (Grosche, 1993; Yahya and Oyewumi, 2013):

$$V_1(\beta) = \tilde{A}_1 \cot^2 \beta + \tilde{B}_1 \frac{\cot \beta}{\sin \beta} \quad (1)$$

$$V_2(\alpha) = \tilde{A}_2 \tan^2 \alpha + \tilde{B}_2 \frac{\tan \alpha}{\cos \alpha} \quad (2)$$

$$V_3(r) = \tilde{A}_3 + \tilde{B}_3 \coth^2 r + \tilde{C}_3 \frac{\coth r}{\sinh r} \quad (1)$$

$$V_4(r) = \tilde{A}_4 + \tilde{B}_4 \tanh^2 r + \tilde{C}_4 \frac{\tanh r}{\cosh r} \quad (2)$$

These kinds of potentials (and their periodic extensions) can be used to describe atomic interactions in solid state Physics (Grosche, 1993; Scarf, 1958; Grosche, 2005). See Yahya and Oyewumi (2013) for other applications of these potentials. Yahya and Oyewumi have obtained the solutions of these potentials for spin- $\frac{1}{2}$ particles using the Nikiforov-Uvarov method.

In this article, the energy eigenvalues and wave functions for the Duffin-Kemmer-Petiau equation in the presence of the Scarf-Grosche potentials are obtained for the case $J = 0$.

The article is organized as follows. In Section 2, we review the improved asymptotic iteration method. The solutions of the DKP equation for the Scarf-Grosche potentials are obtained in Section 3. Section 4 contains the numerical results of the energy eigenvalues and the conclusion is given in Section 5.

2. THE IMPROVED ASYMPTOTIC ITERATION METHOD

The asymptotic iteration method is proposed to solve the second-order differential equations of the form (Cifti et al, 2003; Boztosun and Karakoc, 2007)

$$y'' = \lambda_0(r)y' + s_0(r)y, \quad (3)$$

where $\lambda_0(r) \neq 0$ and $s_0(r), \lambda_0(r)$ are in $C_\infty(a, b)$. We introduce new forms for $\lambda_0(r)$ and $s_0(r)$ as follows:

$$\lambda_0(r) = -\frac{\tau(r)}{\sigma(r)}, \quad s_0(r) = -\frac{\gamma_n}{\sigma(r)}, \quad (4)$$

where τ and σ are new functions and γ_n is a constant with which the energy is quantized. Substituting equation (6) into equation (5), we obtain

$$y'' = -\frac{\tau(r)}{\sigma(r)}y' - \frac{\gamma_n}{\sigma(r)}y. \quad (5)$$

In order to obtain the energy eigenvalues, the following equation was derived (Boztosun and Karakoc, 2007)

$$\gamma_n = -n\tau'(r) - \frac{n(n-1)}{2}\sigma''(r). \quad (6)$$

This expression gives the energy eigenvalues directly. To obtain the eigenfunction, we use

$$y_n(r) = \frac{1}{\rho(r)} \frac{d^n}{ds^n} [\sigma(r)^n \rho(r)], \quad n = 0, 1, 2, 3, \dots \quad (7)$$

where $\rho(r)$ has to satisfy (Nikiforov et al, 1991; Boztosun and Karakoc, 2007)

$$[\sigma(r)\rho(r)]' = \tau(r)\rho(r). \quad (8)$$

3. THE DUFFIN-KEMMER-PETIAU EQUATION

The stationary states of the first order relativistic Duffin-Kemmer-Petiau equation is written as (Yazarloo et al, 2013; Hassanabadi et al, 2012; Yasuk et al. 2008)

$$(\beta \cdot \vec{p}c + mc^2 + U_s + \beta^0 U_v)\Psi(\vec{r}) = \beta^0 E\Psi(\vec{r}), \quad (9)$$

where the internal variables $\beta^\mu (\mu = 0, 1, 2, 3)$ satisfy the commutation relation

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\nu\lambda} \beta^\mu.$$

$g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $(g^{\mu\nu})^2 = 1$, being the metric tensor of Minkowski space-time.

In the spin-1 particles, β^μ are 10×10 matrices given by

$$\beta^0 = \begin{pmatrix} 0 & \bar{0}\bar{0}\bar{0} \\ \bar{0}^T & 0 & I & 0 \\ \bar{0}^T & I & 0 & 0 \\ \bar{0}^T & 0 & 0 & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0}e_i\bar{0} \\ \bar{0}^T & 0 & I & -is_i \\ -e_i & I & 0 & 0 \\ \bar{0}^T - is_i & 0 & 0 & 0 \end{pmatrix},$$

where s_i are the usual 3×3 spin-1 matrices and $\bar{0} = (0 \ 0 \ 0)$, $e_i = (1 \ 0 \ 0)$, $e_2 = (0 \ 1 \ 0)$ and $e_3 = (0 \ 0 \ 1)$, while I and 0 designate the 3×3 identity and zero matrices respectively (Hassanabadi et al, 2012; Yasuk et al, 2008). U_s and U_v^0 represent the scalar and vector interactions respectively (Yazarloo et al, 2013). Also

$$\Psi(\bar{r}) = \begin{pmatrix} \psi_{upper} \\ i\psi_{lower} \end{pmatrix},$$

and the upper and lower components, respectively, are

$$\psi_{upper} \equiv \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \psi_{lower} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}.$$

For equal scalar and vector potentials, equation (5) can be written after some manipulations as (Yazarloo et al, 2013; Zarrinkamar et al, 2011; Yasuk et al 2008)

$$\left[\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} + (E_{n,J} - U_v^0)^2 - (m + U_s)^2 \right] F_{n,J}(r) = 0, \quad (10)$$

where $\frac{J(J+1)}{r^2}$ is the total angular momentum centrifugal term. When $J = 0$ (i.e. s-wave), equation (12) can then be written as

$$\left[\frac{d^2}{dr^2} + (E_n^2 - m^2) - 2(E_n + m)U_v^0 \right] F_n(r) = 0. \quad (11)$$

Equation (13) can now be solved for the four Scarf-Grosche potentials using the improved asymptotic iteration method.

3.1 Solutions of the DKP equation for the first potential

Substituting the potential in equation (1) into equation (13), we obtain

$$\frac{d^2\Psi(\beta)}{d\beta^2} + \left[\varepsilon - A \frac{\cos^2\beta}{\sin^2\beta} - B \frac{\cos\beta}{\sin^2\beta} \right] \Psi(\beta) = 0, \quad (12)$$

where $(m + E_n)\tilde{A} = A, (m + E_n)\tilde{B} = B, \varepsilon = [E_n^2 - m^2]$.
Substituting

$$s = \cos\beta$$

into equation (14) yields

$$\frac{d^2\Psi(s)}{ds^2} - \frac{s}{1-s^2} \frac{d\Psi(s)}{ds} + \frac{1}{(1-s^2)^2} [\varepsilon(1-s^2) - As^2 - Bs] \Psi(s) = 0. \quad (13)$$

We propose the following ansatz

$$\Psi = (1-s)^{\frac{2\sqrt{A+B+\frac{1}{4}}+1}{4}} (1+s)^{\frac{2\sqrt{A-B+\frac{1}{4}}+1}{4}} \Lambda(s). \quad (14)$$

Substituting equation (16) into equation (15), we obtain $[3 - 4A + 2\sqrt{1+4A+4B} + 2\sqrt{1+4A-4B} + \sqrt{1+4A+4B}\sqrt{1+4A-4B} - 8\varepsilon]\Lambda(s) + 4[-\sqrt{1+4A-4B} + \sqrt{1+4A+4B} + (4 + \sqrt{1+4A-4B} + \sqrt{1+4A+4B})s]\Lambda'(s) + 8(-1+s^2)\Lambda''(s) = 0$. (15)

Comparing equation (17) with equation (7), we obtain $\sigma(s) = 8(-1+s^2)$,

$$\begin{aligned} \tau(s) = 4 & [-\sqrt{1+4A-4B} + \sqrt{1+4A+4B} \\ & + (4 + \sqrt{1+4A-4B} + \sqrt{1+4A+4B})s], \\ \gamma_n = & 3 - 4A + 2\sqrt{1+4A-4B} + 2\sqrt{1+4A+4B} + \\ & \sqrt{1+4A-4B}\sqrt{1+4A+4B} - 8\varepsilon. \end{aligned} \quad (16)$$

Substituting equation (18) into equation (8), we have

$$\begin{aligned} \varepsilon = \frac{1}{8} & [3 - 4A + 2\sqrt{1+4A-4B} + 2\sqrt{1+4A+4B} \\ & + \sqrt{1+4A-4B}\sqrt{1+4A+4B}] + n + \frac{n}{2}\sqrt{1+4A-4B} \\ & + \frac{n}{2}\sqrt{1+4A+4B} + n^2. \end{aligned}$$

The energy eigenvalues for the first Scarf-Grosche potential is then obtained as

$$E_n = \left[\frac{3}{8} - \frac{A}{2} + \frac{x}{2} + \frac{y}{2} + \frac{xy}{2} + n(n+x+y+1) + m^2 \right]^{\frac{1}{2}}, \quad (17)$$

$$\text{where } x = \sqrt{A+B+\frac{1}{4}}, y = \sqrt{A+B+\frac{1}{4}}. \quad (18)$$

To obtain the eigenfunctions, we make use of equations (9), (10), and (18), we obtain $\rho(s)$ as

$$\rho(s) = (-1)^{x+y+1} (1-s)^{x+1} (1+s)^{y+1} \quad (19)$$

Substituting equation (21) into equation (9), we obtain

$$y_n(s) = (-1)^n C_n (1-s)^{-x-1} (1+s)^{-y-1} \frac{d^n}{ds^2} [(1-s^2)^n (1-s)^{x+1} (1+s)^{y+1}], \quad (20)$$

which is the Jacobi polynomial as (Nikiforov and Uvarov, 1988; Gradshteyn and Ryzhik, 2007)

$$y_n(s) = C_n P_n^{(x+1, y+1)}(s).$$

The wave function can then be written as

$$\Psi(\beta) = C_n (1 + \cos\beta)^{\frac{2y+1}{4}} (1 - \cos\beta)^{\frac{2x+1}{4}} P_n^{(x+1, y+1)}(\beta), \quad (21)$$

where C_n is the normalization constant.

3.2 Solutions of the DKP equation for the second potential

The solution of the second Scarf-Grosche potential in equation (2) can be obtained from the solutions of the first Scarf-Grosche potential by making use of the following transformations:

$$\beta \rightarrow \alpha + \frac{\pi}{2}, \tilde{B} \rightarrow -\tilde{B}, B \rightarrow B, x \leftrightarrow y,$$

where $x = \alpha \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$. This potential therefore has the same energy eigenvalues as the first potential i.e.

$$E_n = \left[\frac{3}{8} - \frac{A}{2} + \frac{x}{2} + \frac{y}{2} + \frac{xy}{2} + n(n+x+y+1) + m^2 \right]^{\frac{1}{2}}.$$

By making use of the following transformation

$$\Psi(\beta) = \Psi\left(\beta = \alpha + \frac{\pi}{2}\right),$$

we obtain the wave function of the second potential from the first potential as

$$\Psi(\alpha) = C_n (1 + \sin \beta)^{\frac{2x+1}{4}} (1 - \sin \beta)^{\frac{2y+1}{4}} P_n^{(y+1, x+1)}(\sin \alpha), \quad (22)$$

where C_n is the normalization constant.

3.3 Solutions of the DKP equation for the third potential

Substituting the third Scarf-Grosche potential in equation (3) into equation (13) yields

$$\frac{d^2 \Psi(r)}{dr^2} + \left[\varepsilon - B \coth^2 r - C \frac{\coth r}{\sinh r} \right] \Psi(r) = 0, \quad (23)$$

where

$$(m + E_n) \tilde{A} = A, (m + E_n) \tilde{B} = B, (m + E_n) \tilde{C} = C, \varepsilon = [E_n^2 - m^2].$$

With

$$s = \coth r, \quad 0 \leq r \leq \pi$$

Equation (25) can be written as

$$\frac{d^2 \Psi(s)}{ds^2} + \frac{s}{s^2-1} \frac{d\Psi(s)}{ds} + \frac{1}{(s^2-1)^2} [(\varepsilon - A)(s^2 - 1) - Bs^2 - Cs] \Psi(s) = 0. \quad (24)$$

We propose the wave function ansatz

$$\Psi = (s-1)^{\frac{2\sqrt{B+C+\frac{1}{4}}+1}{4}} (s+1)^{\frac{2\sqrt{B-C+\frac{1}{4}}+1}{4}} \Lambda(s). \quad (25)$$

After substituting equation (27) into equation (26) and applying the improved asymptotic iteration method in section 2, we obtain the energy eigenvalues for the third Scarf-Grosche potential as

$$E_n = \left[A + \frac{B}{2} - \frac{3}{8} - \frac{\alpha}{2} - \frac{\beta}{2} - \frac{\alpha\beta}{2} - n(n + \alpha + \beta + 1) + m^2 \right]^{\frac{1}{2}}. \quad (26)$$

The wave function is also obtained as

$$\Psi(r) = C_n (\cosh r - 1)^{\frac{2\alpha+1}{4}} (\cosh r + 1)^{\frac{2\beta+1}{4}} P_n^{(\alpha+1, \beta+1)}(\cosh r), \quad (27)$$

where

$$\alpha = \sqrt{B + C + \frac{1}{4}}, \beta = \sqrt{B - C + \frac{1}{4}}.$$

3.4 Solutions of the DKP equation for the fourth potential

By making use of the transformations

$$r \rightarrow r + i\frac{\pi}{2}, \quad C \rightarrow iC,$$

the results of the fourth Scarf-Grosche potential can be deduced from the results of the Scarf-Grosche potential. Under this transformation,

$$\sinh r \rightarrow i \cosh r, \cosh r \rightarrow i \sinh r, V_3(r) \rightarrow V_4(r).$$

The energy eigenvalues is then obtained as

$$E_n = \left[A + \frac{B}{2} - \frac{3}{8} - \frac{\alpha}{2} - \frac{\beta}{2} - \frac{\alpha\beta}{2} - n(n + \alpha + \beta + 1) + m^2 \right]^{\frac{1}{2}}, \quad (28)$$

$$\text{where } \alpha = \sqrt{B + iC + \frac{1}{4}}, \beta = \sqrt{B - iC + \frac{1}{4}}$$

The wave function is also deduced from equation (29) as

$$\Psi(r) = C_n (i \sinh r - 1)^{\frac{2\alpha+1}{4}} (i \sinh r + 1)^{\frac{2\beta+1}{4}} P_n^{(\alpha+1, \beta+1)}(i \sinh r). \quad (29)$$

4. NUMERICAL RESULTS

The numerical results of the energy eigenvalues obtained in equations (19), (28) and (30) are given in this Section.

Table 1: The bound state energy levels of the DKP equation for the first or second potential ($\tilde{A} = 3, \tilde{B} = 2, m = 1$).

n	E_n
0	1.87241
1	3.41751
2	4.84225
3	6.20259
4	7.52115
5	8.8098
6	10.0756
7	11.3234
8	12.5562
9	13.7766

10	14.9864
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Table 2: The bound state energy levels of the DKP equation for the third or fourth potential ($\tilde{A} = 3, \tilde{B} = 2, \tilde{C} = 1, m = 20$).

n	E_n
0	22.913
1	22.4445
2	21.9218
3	21.3414
4	20.6993
5	19.9904
6	19.2083
7	18.3448
8	17.3893
9	16.3275
10	15.1396

4. CONCLUSION

We have obtained the solutions of the Duffin-Kemmer-Petiau equation for the four Scarf-Grosche potentials. The energy eigenvalues and wave functions are obtained using the improved asymptotic iteration method. The solutions of the second and fourth potentials are deduced from the first and third respectively. A different method can also be used to obtain the solutions of the DKP equation for the Scarf-Grosche potentials.

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