

Bound State Solutions of the Relativistic Dirac Equation with Equal Scalar and Vector Eckart Potentials Using the Nikiforov-Uvarov Method

W A Yahya¹, K J Oyewumi², C O Akoshile³ and T T Ibrahim⁴

ABSTRACT: The relativistic Dirac equation with equal Eckart scalar and vector potentials is solved using the Nikiforov-Uvarov method. The exact energy equation and the spinor wave function for the s-wave bound states are obtained. A straightforward extension to shifted Hulthén potential is also presented.

KEYWORDS: Dirac equation, Eckart potential, Nikiforov-Uvarov Method, spinor wavefunctions, Eigenvalues, Shifted Hulthén potential.

I. INTRODUCTION

It is well known that when a particle is in a strong potential field, the relativistic effect must be considered, leading to the relativistic quantum mechanical description of such particle [1-4]. In the relativistic limit, the particle motions are commonly described using either the Klein-Gordon equation or the Dirac equation [2, 3] depending on the spin character of the particle. The spin-zero particles for example, the mesons, are described by the Klein-Gordon equation. On the other hand, however, the spin-half particles, such as an electron, are satisfactorily described by the Dirac equation. Because the exact solutions of these equations with physically relevant potentials play an important role in our understanding of the relativistic phenomena, attentions have therefore been devoted to obtaining their analytical solutions using different potential models [5-14].

In recent time, a number of researchers have studied the bound states of the Klein-Gordon equation and the Dirac equation with mixed potentials by assuming equal scalar and vector potentials. For example, these investigations have employed anharmonic oscillator potential [4], Kratzer potential [8, 15], Hulthén potential [16], Woods-Saxon potential [17], the Pöschl-Teller potential and Rosen-Morse potential [18], and the harmonic oscillator plus a novel angle-dependent potential [19]. Different methods such as the supersymmetry, asymptotic iteration method and the Nikiforov-Uvarov technique, etc, have been used to solve the differential equations arising from these considerations.

In this paper, our focus is on the bound state solutions of the Dirac equation with equal scalar and vector potentials for the Eckart potential using the elegant Nikiforov-Uvarov (NU) method [20].

^{1,2,3} Theoretical Physics Section, Department of Physics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria.. **062410A** Sept. 2010
E-mails: wazzy4real@yahoo.com, mjphysics@yahoo.com

⁴ Department of Physics, University of Stellenbosch, Matieland, Private Bag XI. Stellenbosch, South Africa.
Email: ttibrahimng@yahoo.com

The Eckart potential, introduced by Eckart [21] in 1930, is a diatomic molecular potential model widely used in applied Physics [22] and chemical Physics [23, 24].

II. NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov method reduces the second order differential equations (ODEs) to the hypergeometric type with an appropriate coordinate transformation $s=s(r)$

$$\Psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \Psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \Psi(s) = 0, \quad (1)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a polynomial of first degree [20, 25-28]. By using the following factorization

$$\Psi(s) = \phi(s)y(s). \quad (2)$$

Eq. (1) reduces to the hypergeometric form [20]

$$\sigma(s)y''(s) + \tau(s)y'(s) + \Lambda y(s) = 0, \quad (3)$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad (4)$$

and $\pi(s)$ is related to the function $\phi(s)$ by

$$\pi(s) = \sigma(s) \frac{d}{ds} (\ln \phi(s)). \quad (5)$$

The parameter Λ is defined by ($\Lambda = \Lambda_n$)

$$\Lambda_n + n\tau' + \left[\frac{n(n-1)\sigma''}{2} \right] = 0; n=0, 1, 2. \quad (6)$$

The energy eigenvalues can be calculated from Eq. (6). In order to calculate the energy eigenvalues, we need, first, to determine Λ by using the first derivative of $\pi(s)$ and defining

$$\kappa = \Lambda - \pi'(s). \quad (7)$$

By solving the resulting quadratic equation for $\pi(s)$, one obtains the following expression

$$\pi(s) = \left(\frac{\sigma' - \tilde{\tau}}{2} \right) \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + \kappa\sigma}. \quad (8)$$

Here $\pi(s)$ is a polynomial and the prime denote the first derivative of the functions $\sigma(s)$ and $\tau(s)$, respectively. The wave function can now be determined by solving for $\phi(s)$ in Eq. (5), and using the Rodrigues relation

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s) \rho(s)], \quad (9)$$

where C_n is a normalization constant and the weight function $\rho(s)$ satisfies the following relation

$$\frac{d}{ds} [\sigma(s) \rho(s)] = \tau(s) \rho(s). \quad (10)$$

III. BOUND STATE SOLUTIONS OF THE DIRAC EQUATION

The Dirac equation with both the scalar potential $S(r)$ and vector potential $V(r)$ is given by ($\hbar = c = 1$) [5]

$$\{\boldsymbol{\alpha} \cdot \mathbf{p} + V(r) + \beta[M + S(r)]\} \Psi(r) = E \Psi(r), \quad (11)$$

where E denotes the particle energy and M is the particle mass. In the relativistic quantum mechanics, the complete set of the conservative quantities for a particle in a central field can be taken as (H, K, J^2, J_z) , the eigenfunctions of which are given by [28];

$$\Psi(r) = \frac{1}{r} \begin{bmatrix} F(r) \phi_{jmj}^A \\ iG(r) \phi_{jmj}^B \end{bmatrix} \quad (k = j + \frac{1}{2}), \quad (12a)$$

$$\Psi(r) = \frac{1}{r} \begin{bmatrix} F(r) \phi_{jmj}^B \\ iG(r) \phi_{jmj}^A \end{bmatrix} \quad (k = -j - \frac{1}{2}). \quad (12b)$$

In Eqs. (12a) and (12b), $F(r)/r$ and $G(r)/r$ are the radial wavefunctions and, ϕ_{jmj}^A and ϕ_{jmj}^B are the angular components of the total wavefunctions which are usually expressed in terms of the spherical harmonics. Substituting Eq. (12a) or (12b) into Eq. (11), gives the radial part of the Dirac equation as a system of the two coupled first order differential equations

$$\frac{dF}{dr} - \frac{k}{r} F = [M + E + S(r) - V(r)] G, \quad (13a)$$

$$\frac{dG}{dr} + \frac{k}{r} G = [M - E + S(r) + V(r)] F. \quad (13b)$$

In this paper, we consider the Eckart potential of the form [21]

$$V(r) = V_1 \operatorname{cosech}^2 \alpha r - V_2 \coth \alpha r. \quad (14)$$

Here, V_1 and V_2 describe the depths of the potential well, and the parameter α is related to the range of the potential. In order to obtain an exact analytical solution for the coupled equations with the Eckart scalar potential $S(r)$ and vector potential $V(r)$, we consider only the special case with scalar and vector potentials of equal magnitudes, $V(r) = S(r)$ so that Eq. (13) reduces to the following

$$\frac{dF}{dr} - \frac{k}{r} F = (M + E)G, \quad (15a)$$

$$\frac{dG}{dr} + \frac{k}{r} G = [M - E + 2(V_1 \operatorname{cosech}^2 \alpha r - V_2 \coth \alpha r)]F, \quad (15b)$$

where we have used Eq. (14) for the potentials.

Combining Eqs. (15a) and (15b), we obtain a second order Schrödinger-like equation for the s-wave with orbital angular momentum $l=0$ (i.e. $k=1$ for the Dirac equation)

$$\left[-\frac{d^2}{dr^2} + 2(M + E)(V_1 \operatorname{cosech}^2 \alpha r - V_2 \coth \alpha r) \right] F(r) = \tilde{E} F(r), \quad (16)$$

where we have used $\tilde{E} = E^2 - M^2$ for brevity. Using

$$s = \coth \alpha r, \quad (17)$$

we have

$$\frac{d^2}{dr^2} = 2\alpha^2 \operatorname{cosech}^2 \alpha r \coth \alpha r \frac{d}{ds} + \alpha^2 (\operatorname{cosech}^2 \alpha r)^2 \frac{d^2}{ds^2}, \quad (18)$$

hence Eq. (16) reduces to

$$F''(s) + \frac{2s}{s^2 - 1} F'(s) + \frac{1}{(s^2 - 1)^2} (\varepsilon - \beta(s^2 - 1) + \gamma s) F(s) = 0. \quad (19)$$

In Eq. (19) we have used the following notations

$$\varepsilon = \frac{\tilde{E}}{\alpha^2}, \beta = \frac{A}{\alpha^2}, \gamma = \frac{B}{\alpha^2}, A = 2(M + E)V_1 \text{ and } B = 2(M + E)V_2. \quad (20)$$

By comparing Eq. (19) with Eq. (1), we obtain $\sigma(s) = s^2 - 1$, $\tilde{\tau}(s) = 2s$ and $\tilde{\sigma}(s) = \varepsilon - \beta(s^2 - 1) + \gamma s$.

Substituting these polynomials into Eq. (8), we obtain the following:

$$\pi(s) = \begin{cases} \pm s(-\frac{\varepsilon}{2} + \frac{\omega}{2}) - \frac{\gamma}{2}, & \text{for } \kappa = -\beta - \frac{\varepsilon}{2} + \frac{\omega}{2} \\ \pm s(-\frac{\varepsilon}{2} - \frac{\omega}{2}) - \frac{\gamma}{2}, & \text{for } \kappa = -\beta - \frac{\varepsilon}{2} - \frac{\omega}{2} \end{cases}, \quad (21)$$

where $\omega = \sqrt{\varepsilon^2 - \gamma^2}$.

For the polynomial $\tau = \tilde{\tau} + 2\pi$, with negative derivative, we select

$$\pi = +s(-\frac{\varepsilon}{2} - \frac{\omega}{2}) - \frac{\gamma}{2} \quad \text{and} \quad k = -(\beta + \frac{\varepsilon}{2}) - \frac{\omega}{2}, \quad (22)$$

which if combined with $\Lambda = k + \pi'(s)$, gives

$$\tau = s(2 - \varepsilon - \omega) - \gamma, \quad (23)$$

and

$$\Lambda = -\beta - \varepsilon - \omega. \quad (24)$$

Now using Eq. (6), we have

$$\Lambda = n(\varepsilon + \omega - 2) - n(n-1). \quad (25)$$

Comparing Eqs. (24) and (25), the exact energy eigenvalues are obtained as

$$\tilde{E} = \alpha^2 \left[\frac{n^2 + n - n\omega - \beta - \omega}{n+1} \right],$$

or more explicitly,

$$\tilde{E} = \alpha^2 \left[\frac{n^2 + n - n(\varepsilon^2 - \gamma^2)^{\frac{1}{2}} - \beta - (\varepsilon^2 - \gamma^2)^{\frac{1}{2}}}{n+1} \right], \quad (26)$$

since $\omega = (\varepsilon^2 - \gamma^2)^{\frac{1}{2}}$. Then, solving for the function $\phi(s)$ by using $\sigma(s) = s^2 - 1$, and Eq. (22) in Eq. (5), we have

$$\phi(s) = (s^2 - 1)^{-\frac{1}{4}(\varepsilon + \omega)} \left(\frac{s-1}{s+1} \right)^{-\frac{\gamma}{4}}. \quad (27)$$

Similarly by using Eqs. (9) and (10), we obtain

$$y_n(s) = C_n (1+s)^{-x} (1+s)^{-y} \frac{d^n}{ds^n} \left[(1-s^2)^n (1-s)^x (1+s)^y \right]. \quad (28)$$

The solutions of Eq. (28) may be expressed in terms of Jacobi polynomials of the form [29]

$$y_n \equiv P_n^{(x,y)}(s) \quad (29)$$

Therefore the radial wavefunction of the upper component of the Dirac wavefunction $F(r)$ can be written as

$$F(r) = C_n \left(\coth^2 \alpha r - 1 \right)^{-\frac{1}{4}(\varepsilon + \omega)} \left(\frac{\coth \alpha r - 1}{\coth \alpha r + 1} \right)^{-\frac{\gamma}{4}} P_n^{(x,y)}(\coth \alpha r), \quad (30)$$

$$\text{where } x = -\left(\frac{\varepsilon + \omega + \gamma}{2} \right) \text{ and } y = -\left(\frac{\varepsilon + \omega - \gamma}{2} \right), \text{ and } \omega = (\varepsilon^2 - \gamma^2)^{\frac{1}{2}}.$$

The normalization constant C_n is obtained as

$$C_n = \sqrt{\frac{n!(-\varepsilon - \omega + 2n + 1) \Gamma\left(-\frac{\varepsilon}{2} - \frac{\omega}{2} + \frac{\gamma}{2} + n + 1\right)}{2^{(-\varepsilon - \omega + 2n + 1)} \Gamma\left(-\frac{\varepsilon}{2} - \frac{\omega}{2} - \frac{\gamma}{2} + n + 1\right) \Gamma\left(-\frac{\varepsilon}{2} - \frac{\omega}{2} + \frac{\gamma}{2} + n + 1\right)}}. \quad (31)$$

The lower component $G(r)$ may be defined from Eq. (15a) as $G(r) = \frac{\frac{dF}{dr} - \frac{k}{r} F}{M + E}$, hence

$$G(r) = \frac{1}{M + E} \left\{ e^{\gamma \alpha r} (\sinh \alpha r)^{\left(\frac{\varepsilon + \omega}{2}\right)} P_n^{(x,y)}(\coth \alpha r) \left[\alpha r + \frac{\alpha(\varepsilon + \omega)}{2} \coth \alpha r - \frac{k}{r} \right] + \right. \\ \left. e^{\gamma \alpha r} (\sinh \alpha r)^{\frac{\varepsilon + \omega}{2}} \left(\frac{n - \omega - \varepsilon + 1}{2} \right) P_{n-1}^{(x+1,y+1)}(\coth \alpha r) \right\} \quad (32)$$

Finally, using the radial wavefunctions, Eqs. (30) and (32), in Eqs. (12a) and (12b) gives the s-wave spinor wavefunctions for the Dirac equation with equally mixed Eckart potential.

IV. DISCUSSION

In this section, within the framework of the Dirac theory with equally mixed potentials, we obtain the energy equation and the spinor wave functions for the Hulthén potential, a special case of the Eckart potential. If we take $\alpha = \frac{1}{2a}$, $V_1 = 0$ and $V_2 = \frac{V_0}{2}$, the Eckart potential given in Eq. (14)

reduces to the standard Hulthén potential with a constant shift of $\frac{V_0}{2}$ [30],

$$V(r) = -V_0 \frac{e^{-r/a}}{1 - e^{-r/a}} - \frac{V_0}{2}. \quad (33)$$

The Hulthén potential is widely used in many branches of physics, such as atomic physics [31], solid state physics [32] and chemical physics [33]. Using Eq. (16) and making the corresponding parameter replacements in Eq. (26), we obtain the energy eigenvalue equation for the Hulthén potential in the Dirac theory with equally mixed potentials,

$$\tilde{E} = \frac{1}{4a^2} \left[\frac{n^2 + n - n\omega - \beta - \omega}{n+1} \right]. \quad (34)$$

Making the corresponding parameter replacements in Eqs. (30) and (32), we can consequently obtain the wavefunctions $F(r)$ and $G(r)$, respectively.

$$F(r) = C_n \left(\coth^2 \frac{1}{2a} r - 1 \right)^{-\frac{1}{4}(\varepsilon + \omega)} \left(\frac{\coth \frac{1}{2a} r - 1}{\coth \frac{1}{2a} r + 1} \right)^{-\frac{\gamma}{4}} P_n^{(x,y)} \left(\coth \frac{1}{2a} r \right), \quad (35)$$

$$G(r) = \frac{1}{M + E} \left\{ e^{\frac{\gamma}{2a} r} \left(\sinh \frac{1}{2a} r \right)^{\left(\frac{\varepsilon + \omega}{2}\right)} P_n^{(x,y)} \left(\coth \frac{1}{2a} r \right) \left[\frac{1}{2a} r + \frac{(\varepsilon + \omega)}{4a} \coth \frac{1}{2a} r - \frac{k}{r} \right] + e^{\frac{\gamma}{2a} r} \left(\sinh \frac{1}{2a} r \right)^{\frac{\varepsilon + \omega}{2}} \left(\frac{n - \omega - \varepsilon + 1}{2} \right) P_{n-1}^{(x+1,y+1)} \left(\coth \frac{1}{2a} r \right) \right\} \quad (36)$$

where $x = -\left(\frac{\varepsilon + \omega + \gamma}{2}\right)$ and $y = -\left(\frac{\varepsilon + \omega - \gamma}{2}\right)$, and $\omega = (\varepsilon^2 - \gamma^2)^{\frac{1}{2}}$.

V. CONCLUSION

The bound state solutions of the Dirac equation with equally mixed Eckart potential have obtained. The two-component spinor eigenfunctions (which are obtained in terms of the Jacobi polynomials) and energy eigenvalues for the s-wave ($k=1$) have been obtained using the Nikiforov-Uvarov method. It is evident that this method can be used also to find the bound state solutions for all l (orbital angular momentum).

REFERENCES

1. Wang, I.C., Wong, C.Y. (1988): Phys. Rev. D **34**, 348
2. Yi, L.Z., Diao, Y.F., Liu, J.Y., Jia, C.S. (2004): Phys Lett. A **333**, 212-217
3. Zhang, X.C., Liu, Q.W., Jia, C.S., Wang, L.Z. (2005): Phys Lett. A **340**, 59-69
4. Wei, G.F., Long, C.Y., He, Z., Qin, S.J. (2007): Phys. Scr. **76**, 442
5. Su, R.K., Ma, Z.Q. (1986): J. Phys. A: Math. Gen. **19**, 1739
6. Dominguez-Adame, F. (1989): Phys. Lett. A **136**, 175

7. Qiang, W.C. (2003): Chin. Phys. **12**, 136
8. Qiang, W.C. (2004): Chin. Phys. **13**, 575
9. Chen, G., Chen, Z.D., Liu, Z.M. (2004): Chin. Phys **13**, 279
10. Zhang, X.A., Chen, K., Duan, Z.L (2005): Chin. Phys. **14**, 42
11. Diao, Y.F., Yi, L.Z., Jia, C.S. (2004): Phys. Lett. A **332**, 157
12. Yi, L.Z., Diao, Y.F., Liu, J.Y., Jia, C.S. (2004): Phys Lett. A **333**, 212
13. Zhao, X.Q. Jia, C.S., Yang, Q.B. (2005): Phys. Lett. A **337**, 189
14. Zou, X., Yi, L.Z., Jia, C.S. (2005): Phys. Lett. A **346**, 54-64
15. Qiang, W.C. (2003): Chin. Phys. **12**(10), 1054
16. Simsek, M., Egrifes, H. (2004): J. phys. A: Math. Gen. **37**, 4379
17. Berkdemir, C., Berkdemir, A, Sever, R. (2006): J. Phys. A: Math.Gen. **39**, 13455
18. Alhaidari, A. (2001): J. Phys. A: Math. Gen. **34**, 9827
19. Berkdemir, C., Cheng, Y.F. (2009): Phys. Scr. **79**, 035003
20. Nikiforov, A.F., Uvarov, V.B. (1988). Special functions of Mathematical Physics, (Basle, Birkhauser).
21. Eckart, C. (1930): Phys. Rev. **35**, 1303
22. Cooper, F., Khare, A., Sukhatme, U. (1995): Phys. A: Math. Gen. **39**, 7737
23. Weiss, J.J. (1964): J. Chem. Phys. **41**, 1120
24. Cimas, A., Aschi, M., Barrientos, C., Rayon, V.M., Sordo, J.A. and Largo, A. (2003): Chem. Phys. Lett. **374**, 594
25. Egrifes, H., Demirham, D., Buyukkilic, F., (1999): Physica Scripta **59**, 90
26. Egrifes, H., Demirham, D., Buyukkilic, F., (1999): Physica Scripta **60**, 195
27. Buyukkilic, F., Egrifes, H., Demirham, D. (1997): Theor. Chem. Acc. **98**, 192
28. Messiah, A. (2000): Quantum Mechanics (New York: Dover);
Greiner, W. (2000): Relativistic Quantum Mechanics: Wave equations 3rd ed. (Springer, Berlin).
29. Gradshteyn, I.S., Ryzhik I.M. (1994): Tables of Integrals: Series and products, 5th ed. (Academic Press, San Diego).
30. Hulthén, L. (1942): Ark. Mat. Astron, Fys. **A28** 5
31. Malli, G. (1981): Chem. Phys. Lett. **26** 578
32. Dow, J. D. (1973): Phys. Rev. Lett. **30** 903
33. Pyykko, P., Jokisaari, J. 1975: Chem. Phys. **10** 1975