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Bound state solutions of the Dirac equation for the trigonometric and hyperbolic Scarf-Grosche potentials using the Nikiforov-Uvarov method

W. A. Yahya^{1,2,a)} and K. J. Oyewumi^{1,b)}

¹Theoretical Physics Section, Department of Physics, University of Ilorin, Nigeria

²Department of Physics, Kwara State University, Malete, Ilorin, Nigeria

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The relativistic Dirac equation with equal (trigonometric and hyperbolic) Scarf-Grosche scalar and vector potentials in $3 + 1$ and $1 + 1$ space-time dimension are solved by using the Nikiforov-Uvarov method. The exact energy equations and the spinor wave functions are obtained for these sets of potentials. The bound states energy are calculated numerically. Also, with appropriate transformations, solutions of the fourth potential V_4 can be deduced from the solutions of the third potential V_3 .

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I. INTRODUCTION

When a particle is in a strong potential field, the relativistic effect must be considered leading to the relativistic quantum mechanical description of such particle. When we consider this condition, a particle in the strong potential field should be described by either the Klein-Gordon equation, the Dirac equation, or Duffin-Kemmer-Petiau equations.¹⁻⁷

Since the exact solutions of the Dirac equation with any potential play an important role in relativistic quantum mechanics, a considerable amount of works have been discussed about the Dirac equation with physical potentials by using different methods.^{6,8-21}

By using different methods, some authors studied the bound states solutions of the Klein-Gordon equation and the Dirac equation with mixed typical potentials under the condition that each of the scalar potentials is equal to its vector potential. These investigations include: Scarf-type potential,^{9,12} Eckart potential,^{14,19} Coulomb potential,¹⁶ a new anharmonic potential,²² Kratzer potential,²³ Hulthén potential,^{24,25} Woods-Saxon potential,^{26,27} Morse potential,²⁷ Pöschl-Teller potential,²⁸ Rosen-Morse potential,^{28,29} q-deformed hyperbolic Pöschl-Teller potential³⁰ and the new ring-shaped non-spherical harmonic oscillator potential.³¹

The methods include the standard method, supersymmetry and shape invariance potential, asymptotic iteration method, and the Nikiforov-Uvarov method. The Nikiforov-Uvarov method has been a very useful method in obtaining the explicit exact bound-state solutions of the Schrödinger, Klein-Gordon, Duffin-Kemmer-Petiau, and Dirac equations for 3 or D dimensions for different cases (for $\ell = 0$ or $\ell \neq 0$ that is, including the centrifugal term) with different potentials of interest.^{32,33}

This method has served as a useful tool in various ways as exemplified by different authors. This method gives a systematical framework in obtaining the energy eigenvalues and the corresponding eigenfunctions for the Schrödinger, Klein-Gordon, Duffin-Kemmer-Petiau and Dirac (spin and pseudospin symmetry inclusive) equations for various potentials. Hence, this method is generally applicable to various studies in the fields of mathematical physics, chemical physics, molecular physics, nuclear physics, and other related fields of studies.^{8,20,25,33-51}

a) wazzy4real@yahoo.com.

b) Also at International Chair of Mathematical Physics and Applications (ICMPA), Université d'Abomey-Calavi, Cotonou, Republic of Benin. E-mail: kjoyewumi66@unilorin.edu.ng.

In this paper, we obtain the bound state solutions of the Dirac equation with equal scalar and vector potentials for the trigonometry and hyperbolic Scarf-Grosche potentials in $3 + 1$ and $1 + 1$ space-time dimension using an elegant and simple method called “the Nikiforov-Uvarov method”. By the Scarf-Grosche potentials, we refer to the sets of potentials (trigonometry and hyperbolic) earlier described by Grosche (though, he called them “Scarf-like potentials”).^{52,53}

We prefer to call them “Scarf-Grosche potentials” to distinguish them from other forms of the Scarf-like or Scarf-type potential, earlier discussed by Motavali,⁹ Motavalli and Akbarieh,⁴⁸ and Zhang *et al.*¹² The potentials under discussion as earlier proposed and described by Grosche⁵² are

$$V_1(\beta) = \tilde{A}_1 \cot^2 \beta + \tilde{B}_1 \frac{\cot \beta}{\sin \beta}, \quad (1)$$

$$V_2(\alpha) = \tilde{A}_2 \tan^2 \alpha + \tilde{B}_2 \frac{\tan \alpha}{\cos \alpha}, \quad (2)$$

$$V_3(r) = \tilde{A}_3 + \tilde{B}_3 \coth^2 r + \tilde{C}_3 \frac{\coth r}{\sinh r}, \quad (3)$$

$$V_4(x) = \tilde{A}_4 + \tilde{B}_4 \tanh^2 x + \tilde{C}_4 \frac{\tanh x}{\cosh x}. \quad (4)$$

The first and third Scarf-Grosche potentials $V_{1,3}$ are more singular potentials as the Manning-Rosen-like potential $V^{MR(x)} = -\tilde{A}_1 \cot x + \tilde{B}_1 \csc^2 x$, ($0 < x < \pi$) and V_2 is just V_1 shifted by $\frac{\pi}{2}$.⁵² These latter potentials (and their periodic extensions) can be used to describe atomic interactions in solid-state physics.^{52,54}

The third hyperbolic Scarf-Grosche potential V_3 can be seen as a screened $\frac{1}{r^2}$ potential. Also, it has the additional advantage that there exist bound states, similarly as for $V_{1,2}$. The fourth hyperbolic Scarf-Grosche potential, V_4 , can serve as a model for barrier tunneling with bound and scattering states,⁵⁵ and as the q -deformed hyperbolic barrier potential as earlier described by Pertsch.⁵⁵ Again, it appears in the study of a constant magnetic field on the single-sheeted hyperboloid.⁵² The path integral solutions of the Schrödinger equation of these potentials have been obtained by Grosche.⁵² Also, Grosche has obtained the path integral solutions of the Schrödinger equation of the q -deformed form of the fourth potential V_4 and other related potentials.⁵⁶

The paper is organized as follows. In Sec. II, we review the Nikiforov-Uvarov method. Section III contains the solutions of the Dirac equation for the Scarf-Grosche potentials in $3 + 1$ and $1 + 1$ space-time dimension. Section IV contains the numerical results and the conclusion is given in Sec. V.

II. NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov method^{32,33} reduces the second order differential equations (ODEs) to the hypergeometric type with an appropriate coordinate transformation $s = s(r)$

$$\Psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \Psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \Psi(s) = 0, \quad (5)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a polynomial of first degree^{8,16,18,20,24,25,33–37,40,41,43,47,49,50,56–66}

By using the following factorization:

$$\psi(s) = \phi(s)y(s), \quad (6)$$

Eq. (1) reduces to the hypergeometric form^{32,33}

$$\sigma(s)y''(s) + \tau(s)y'(s) + \Lambda y(s) = 0, \quad (7)$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \quad (8)$$

satisfies the condition $\tau'(s) < 0$, which will have a negative derivative and $\pi(s)$ is related to the function $\phi(s)$ by

$$\pi(s) = \sigma(s) \frac{d}{ds} [\ln \phi(s)]. \quad (9)$$

The parameter Λ is defined by ($\Lambda = \Lambda_n$)

$$\Lambda_n + n\tau' + \left[\frac{n(n-1)}{2} \sigma'' \right] = 0; n = 0, 1, 2. \quad (10)$$

The energy eigenvalues can be calculated from Eq. (10). In order to calculate the energy eigenvalues, first, we need to determine Λ by using the first derivative of $\pi(s)$ and defining

$$K = \Lambda - \pi'(s). \quad (11)$$

By solving the resulting quadratic equation for $\pi(s)$, one obtains the following expression:

$$\pi(s) = \left(\frac{\sigma' - \tilde{\tau}}{2} \right) \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + K\sigma}. \quad (12)$$

Here $\pi(s)$ is a polynomial and the prime denote the first derivative of the functions $\sigma(s)$ and $\tau(s)$, respectively. The wave function can now be determined by solving for ϕ in Eq. (5), and by using the Rodrigues relation⁶⁷

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s) \rho(s)], \quad (13)$$

where C_n is a normalization constant and the weight function $\rho(s)$ satisfies the following relation:

$$\frac{d}{ds} [\sigma(s) \rho(s)] = \tau(s) \rho(s). \quad (14)$$

III. BOUND STATE SOLUTION OF THE DIRAC EQUATION

In the presence of a scalar potential $S(r)$ and vector potential $V(r)$, the Dirac equation with both the scalar potential $S(r)$ and vector potential $V(r)$ can be written as ($\hbar = c = 1$)^{1,3,6,16,18,20,28,48–50,67–70}

$$\{\vec{\alpha} \cdot \mathbf{P} + V(r) + \beta[M + S(r)]\} \Psi(r) = E \Psi(r), \quad (15)$$

where E denotes the particle energy and M is the particle mass. The $\vec{\alpha}$ and β are 4×4 matrices which, in the usual representation, take the form

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad (16)$$

where $\vec{\sigma}$ is the vector Pauli's 2×2 matrices and $\mathbf{I}$ is a 2×2 unit matrix. In the relativistic quantum mechanics, the complete set of the conservative quantities for a particle in a central field can be taken as (H, K, J_2, J_z) , the eigenfunctions of which are given by:¹⁴

$$\Psi(r) = \frac{1}{r} \begin{bmatrix} F(r) \phi_{jmj}^A \\ i G(r) \phi_{jmj}^B \end{bmatrix}, \quad (17)$$

$$\Psi(r) = \frac{1}{r} \begin{bmatrix} F(r) \phi_{jmj}^B \\ i G(r) \phi_{jmj}^A \end{bmatrix}, \quad (18)$$

where $F(r)$ and $G(r)$ are the radial wave functions, and ϕ_{jmj}^A and ϕ_{jmj}^B are the angular components of the total wave functions which are usually expressed in terms of the spherical harmonics. Substituting

Eqs. (17) and (18) into Eq. (16), gives the radial part of the Dirac equation as a system of the two coupled first order differential equations

$$\frac{dF(r)}{dr} + \frac{\kappa}{r} F(r) = [M + E + S(r) - V(r)] G(r), \quad (19)$$

$$\frac{dG(r)}{dr} - \frac{\kappa}{r} G(r) = [M - E + S(r) + V(r)] F(r), \quad (20)$$

where $\kappa = \pm(j + \frac{1}{2})$, for $\ell = j \pm \frac{1}{2}$.

Equations (19) and (20) are the radial parts of the Dirac equation, by considering a special case where the scalar and vector potentials are equal, we have

$$\frac{dF(r)}{dr} + \frac{\kappa}{r} F(r) = [M + E] G(r) \quad (21)$$

and

$$\frac{dG(r)}{dr} - \frac{\kappa G(r)}{r} = [M - E + 2V(r)] F(r). \quad (22)$$

From Eq. (21), we have the lower component spinor $G(r)$

$$G(r) = \frac{\left[\frac{dF(r)}{dr} + \kappa \frac{F(r)}{r} \right]}{M + E}, \quad (23)$$

then, substituting Eq. (23) into Eq. (21), we have a second order differential equation for the upper component spinor $F(r)$ as follows:

$$\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa + 1)}{r^2} + 2(E + M)V(r) \right] F(r) = \tilde{E} F(r), \quad (24)$$

where $\tilde{E} = E^2 - M^2$. Here, $E \neq -M$, only real positive energy states exist (positive energy solution).

On the other hand, by considering another special case where the scalar and vector potentials are equal with opposite sign, one can eliminate $F(r)$ in Eq. (20). With this, a second-order differential equation for the lower component spinor $G(r)$ can be obtained as follows:

$$\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa - 1)}{r^2} + 2(E - M)V(r) \right] G(r) = \tilde{E} G(r), \quad (25)$$

and the upper component spinor $F(r)$ is obtained as

$$F(r) = \frac{\left[\frac{dG(r)}{dr} - \kappa \frac{G(r)}{r} \right]}{M - E}, \quad (26)$$

where $\tilde{E} = E^2 - M^2$. Here, $E \neq M$, only real negative energy states exist (negative energy solution).

The equal scalar $S(r)$ and vector $V(r)$ coupling potentials have attracted much attention in the literature due to the resulting simplification in the solution of the relativistic problem by Alhaidari *et al.*⁶ Hence, the potential term in the equation above can be scaled so that in the non-relativistic limit, the interaction becomes V and not $2V$.

The discussion above is on the Dirac equation in $3 + 1$ dimensions. Equations (24) and (25) cannot be solved analytically except for $\ell = 0$ or $\kappa = -1$ and ($\bar{\ell} = 0$ or $\kappa = 1$), respectively. Therefore, we choose to solve the problem for the s -wave case, for $V(3, 4)$ potentials by using the Nikiforov-Uvarov method. On the other hand, for potential $V(1, 2)$, the Dirac equation in $1 + 1$ dimension that replaces Eqs. (24) and (25) can be written as follows:⁷

$$\frac{dF(x)}{dx} = [M + E + S(x) - V(x)] G(x), \quad (27)$$

$$\frac{dG(x)}{dx} = [M - E + S(x) + V(x)] F(x). \quad (28)$$

Then, by considering an equally mixed scalar and vector potentials with the same sign, we have the second ordinary differential equation for the upper component spinor

$$\left[-\frac{d^2}{dx^2} + 2(E + M)V(x) \right] F(x) = \tilde{E}F(x), \quad (29)$$

and the lower component spinor $G(x)$, respectively, as

$$G(x) = \frac{1}{M + E} \frac{dF(x)}{dx}, \quad (30)$$

where $\tilde{E} = E^2 - M^2$, $E + M \neq 0$, the positive energy solution.

Similarly, for an equally mixed scalar and vector potentials with the opposite sign, we have the lower and upper component spinors, respectively, as

$$\left[-\frac{d^2}{dx^2} + 2(E - M)V(x) \right] G(x) = \tilde{E}G(x) \quad (31)$$

and

$$F(x) = \frac{1}{M - E} \frac{dG(x)}{dx}, \quad (32)$$

where $\tilde{E} = E^2 - M^2$, $E - M \neq 0$, the negative energy solution. In this paper, we shall obtain the bound state solutions of the sets of the Scarf-Grosche potentials mentioned above.

A. Solutions of the Dirac equation in 1 + 1 dimension for $V_1(\beta) = \tilde{A} \cot^2 \beta + \tilde{B} \frac{\cot \beta}{\sin \beta}$

This potential is defined over the configuration space $0 \leq \beta \leq \pi$. This physical system is confined between the boundaries where $V_1(0)$ and $V_1(\pi)$ become infinite. Therefore, a coordinate transformation $x = \beta(x)$ such that $\beta(0) = 0$, $\beta(\infty) = \pi$ and $\frac{d^2}{dx^2} = \frac{d^2}{d\beta^2}$ are required. Such requirement dictates that the problem can be solved only in 1 + 1 not in 3 + 1 dimensions.

1. Solutions of the positive energy subspace of the Dirac equation in 1 + 1 dimension for $V_1(\beta)$

With this coordinate transformation and the substitution of the potential function $V_1(\beta)$ into Eq. (29), and taking into consideration the Alhaidari's suggestion,⁶ Eq. (29) gives

$$\frac{d^2 F(\beta)}{d\beta^2} + \left[\epsilon - A \frac{\cos^2 \beta}{\sin^2 \beta} - B \frac{\cos \beta}{\sin^2 \beta} \right] F(\beta) = 0; x = \beta \in [0, \pi], \quad (33)$$

where

$$(M + E) \tilde{A} = A, \quad \tilde{E} = \epsilon, \quad (M + E) \tilde{B} = B. \quad (34)$$

Substituting

$$s = \cos \beta; 0 \leq \beta \leq \pi \quad (35)$$

into Eq. (33) to have

$$\frac{d^2 F(s)}{ds^2} - \frac{s}{1-s^2} \frac{dF(s)}{ds} + \frac{1}{(1-s^2)^2} [\epsilon(1-s^2) - As^2 - Bs] F(s) = 0. \quad (36)$$

Comparing Eq. (36) with Eq. (5), we obtain the following relations:

$$\sigma(s) = 1 - s^2, \quad \tilde{\tau}(s) = -s, \quad \sigma'(s) = -2s, \quad \sigma''(s) = -2, \quad \tilde{\sigma}(s) = \epsilon(1 - s^2) - As^2 - Bs. \quad (37)$$

Substituting Eq. (37) into Eq. (12), we obtain

$$\pi(s) = -\frac{s}{2} \pm \sqrt{s^2(\epsilon + A + \frac{1}{4} - K) + Bs + (K - \epsilon)}. \quad (38)$$

In order to obtain K , we equate the discriminate of the square root to zero. We thus obtain

$$K = \epsilon + \frac{A}{2} + \frac{1}{8} \pm \frac{\gamma}{2}, \quad (39)$$

where

$$\gamma = \sqrt{A^2 - B^2 + \frac{A}{2} + \frac{1}{16}}. \quad (40)$$

Then, we have

$$\pi(s) = \begin{cases} -\frac{s}{2} \pm \left[\left(\frac{A}{2} - \frac{\gamma}{2} + \frac{1}{8} \right) s + \frac{B}{2} \right], & \text{for } K = \epsilon + \frac{A}{2} + \frac{1}{8} + \frac{\gamma}{2} \\ -\frac{s}{2} \pm \left[\left(\frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8} \right) s + \frac{B}{2} \right], & \text{for } K = \epsilon + \frac{A}{2} + \frac{1}{8} - \frac{\gamma}{2}. \end{cases} \quad (41)$$

The value of π is chosen such that $\tau'(s)$ has a negative derivative.^{32,33} We choose the values

$$\pi = -\frac{s}{2} - \left(\frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8} \right) s - \frac{B}{2} \text{ and } K = \epsilon + \frac{A}{2} + \frac{1}{8} - \frac{\gamma}{2} \quad (42)$$

and since $\tilde{\tau}(s) = -s$, we have

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) = -2s - s(A + \gamma + \frac{1}{4}) - B, \quad (43)$$

and

$$\tau'(s) = -\frac{9}{4} - A - \gamma. \quad (44)$$

Substituting Eq. (44) with $\sigma''(s) = -2$ into Eq. (10) gives

$$\Lambda = n \left(\frac{9}{4} + A + \gamma \right) + n(n-1) \quad (45)$$

and on substituting Eq. (42) into Eq. (11), we have

$$\Lambda = \epsilon - \gamma - \frac{1}{2}. \quad (46)$$

Equating Eqs. (45) and (46) yields

$$E^2 - M^2 = n^2 + n \left(A + \gamma + \frac{5}{4} \right) + \gamma + \frac{1}{2}, \quad (47)$$

Eq. (47) is the energy equation for the Scarf-Grosche potential $V_1(\beta)$ for the Dirac theory with equally mixed potentials. From Eq. (47), we obtain

$$E_n = \left[n^2 + n \left(A + \gamma + \frac{5}{4} \right) + \gamma + \frac{1}{2} + M^2 \right]^{\frac{1}{2}}, \quad (48)$$

which is the energy eigenvalues.

To obtain the eigenfunctions, we make use of Eqs. (6) and (9), to have

$$\phi(s) = (1 - s^2)^{\frac{\gamma}{2}} \left(\frac{1+s}{1-s} \right)^{-\frac{B}{4}}, \quad (49)$$

where

$$Y = \frac{A}{2} + \frac{\gamma}{2} + \frac{5}{8}. \quad (50)$$

Substituting $\sigma(s) = 1 - s^2$ into Eq. (13), we obtain the polynomial solution

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} \left[(1 - s^2)^n \rho(s) \right]. \quad (51)$$

Also, from Eq. (14) we have

$$\frac{d}{ds} [(1-s^2) \rho(s)] = \left[-s \left(A + \gamma + \frac{9}{4} \right) + B \right] \rho(s), \quad (52)$$

from which $\rho(s)$ can be obtained as

$$\rho(s) = (1-s^2)^{\frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8}} \left(\frac{1+s}{1-s} \right)^{-\frac{B}{2}}. \quad (53)$$

Substituting Eq. (53) into Eq. (51) yields

$$y_n(s) = C_n (1-s)^{-\frac{B}{2} - \frac{A}{2} - \frac{\gamma}{2} - \frac{1}{8}} (1+s)^{-(-\frac{B}{2} + \frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8})} \times \frac{d^n}{ds^n} \left[(1-s^2)^n (1-s)^{\frac{B}{2} + \frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8}} (1+s)^{-\frac{B}{2} + \frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8}} \right], \quad (54)$$

and putting

$$x = \frac{B}{2} + \frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8} \text{ and } y = -\frac{B}{2} + \frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8}, \quad (55)$$

in Eq. (54), we have

$$y_n(s) = C_n (1-s)^{-x} (1+s)^{-y} \frac{d^n}{ds^n} \left[(1-s^2)^n (1-s)^x (1+s)^y \right]. \quad (56)$$

Equation (56) is the Jacobi polynomials, and it can be written as^{71,72}

$$y_n(s) = C_n P_n^{(x,y)}(s), \quad (57)$$

then, $\psi(s)$ can then be written as

$$\psi(s) = C_n (1-s^2)^{\frac{A}{2} + \frac{\gamma}{4} + \frac{5}{16}} \left(\frac{1+s}{1-s} \right)^{-\frac{B}{4}} P_n^{(x,y)}(s), \quad (58)$$

and therefore,

$$F_n(\beta) = C_n (1 + \cos \beta)^{-\frac{B}{4} + \frac{A}{4} + \frac{\gamma}{4} + \frac{5}{16}} (1 - \cos \beta)^{\frac{B}{4} + \frac{A}{4} + \frac{\gamma}{4} + \frac{5}{16}} P_n^{(x,y)}(\cos \beta), \quad (59)$$

where C_n is the normalization constant.

Equation (59) is the upper component of the wave function corresponding to energy level E_n , and on substituting Eq. (59) into Eq. (30), we obtain

$$G_n(\beta) = \frac{C_n}{M + E_n} \left\{ \left[(1 + \cos \beta)^p (1 - \cos \beta)^q P_n^{(x,y)}(\cos \beta) \left(\frac{-p \sin \beta}{(1 + \cos \beta)} + \frac{q \sin \beta}{(1 - \cos \beta)} \right) \right] \right. \\ \left. + \frac{C_n}{M + E_n} \left\{ \frac{(1 + \cos \beta)^p (1 - \cos \beta)^q (n + x + y + 1)}{2} P_{n-1}^{(x+1,y+1)}(\cos \beta) \right\} \right\}, \quad (60)$$

where $x = \frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8} + \frac{B}{2}$, $y = \frac{A}{2} + \frac{\gamma}{2} + \frac{1}{8} - \frac{B}{2}$, $p = \frac{A}{4} + \frac{\gamma}{4} + \frac{5}{16} - \frac{B}{4}$, $q = \frac{A}{4} + \frac{\gamma}{4} + \frac{5}{16} + \frac{B}{4}$, and C_n is the normalization constant.

2. Solutions of the negative energy subspace of the Dirac equation in 1 + 1 dimension for $V_1(\beta)$

Similarly, the solutions of the negative energy subspace of the Dirac equation in 1 + 1 dimension can be obtained by using the coordinate transformation in Sec. III A 1. With this transformation, the resulting equation with the substitution of the potential function $V_1(\beta)$ into Eq. (31) (with the

Alhaidari's suggestion⁶) can be solved by using the Nikiforov-Uvarov method to obtain the following solutions: The energy eigenvalues is obtained as

$$E_n = - \left[n^2 + n \left(A' + \gamma' + \frac{5}{4} \right) + \gamma' + \frac{1}{2} + M^2 \right]^{\frac{1}{2}}, \quad (61)$$

where $A' = (E - M)\tilde{A}$, $B' = (E - M)\tilde{B}$, $\gamma' = \sqrt{A'^2 - B'^2 + \frac{A'}{2} + \frac{1}{16}}$; the lower and upper component spinors are

$$G_n(\beta) = C'_n (1 + \cos \beta)^{-\frac{B'}{4} + \frac{A'}{4} + \frac{\gamma'}{4} + \frac{5}{16}} (1 - \cos \beta)^{\frac{B'}{4} + \frac{A'}{4} + \frac{\gamma'}{4} + \frac{5}{16}} P_n^{(x,y)}(\cos \beta) \quad (62)$$

and

$$F_n(\beta) = \frac{C'_n}{M - E_n} \left\{ \left[(1 + \cos \beta)^p (1 - \cos \beta)^q P_n^{(x,y)}(\cos \beta) \left(\frac{-p \sin \beta}{(1 + \cos \beta)} + \frac{q \sin \beta}{(1 - \cos \beta)} \right) \right] \right. \\ \left. + \frac{C'_n}{M - E_n} \left\{ \frac{(1 + \cos \beta)^p (1 - \cos \beta)^q (n + x + y + 1)}{2} P_{n-1}^{(x+1,y+1)}(\cos \beta) \right\} \right\}, \quad (63)$$

where C'_n is the normalization constant.

B. Solutions of the Dirac equation in 1 + 1 dimension for $V_2(\alpha) = \tilde{A} \tan^2 \alpha + \tilde{B} \frac{\tan \alpha}{\cos \alpha}$

The potential $V_2(\alpha)$ is deformed over the configuration space $-\frac{\pi}{2} \leq \alpha \leq +\frac{\pi}{2}$. This physical system is restricted to the region between the two boundaries where $V_2(-\frac{\pi}{2})$ and $V_2(+\frac{\pi}{2})$ become infinite. Therefore, a coordinate transformation $x = \alpha(x)$ such that $\alpha(0) = -\frac{\pi}{2}$, $\alpha(\infty) = +\frac{\pi}{2}$ and $\frac{d^2}{dx^2} = \frac{d^2}{d\alpha^2}$ are required. Similarly, such requirement dictates that the problem can be solved only in 1 + 1 not in 3 + 1 dimensions.

1. Solutions of the positive energy subspace of the Dirac equation in 1 + 1 dimension for $V_2(\alpha)$

The solution of the potential function $V_2(\alpha)$ in this section is obtained from that of the potential function $V_1(\beta)$ in Sec. III A by using the following transformations: $\beta \rightarrow \alpha + \frac{\pi}{2}$, $\bar{B} \rightarrow -\bar{B}$, $B \rightarrow -B$, $x \leftrightarrow y$, $p \leftrightarrow q$, where $\alpha \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$. This potential has the same energy eigenvalues as that of the potential $V_1(\beta)$ as

$$E_n = \left[n^2 + n \left(A + \gamma + \frac{5}{4} \right) + \gamma + \frac{1}{2} + M^2 \right]^{\frac{1}{2}}. \quad (64)$$

With the transformation above, one can deduce that

$$\Psi_n(\beta) = \Psi_n(\beta = \alpha + \frac{\pi}{2}), \quad (65)$$

and therefore, the upper and lower component spinors can be obtained, respectively, as:

$$F_n(\alpha) = C_n (1 - \sin \alpha)^{\frac{B}{4} + \frac{A}{4} + \frac{\gamma}{4} + \frac{5}{16}} (1 + \sin \alpha)^{-\frac{B}{4} + \frac{A}{4} + \frac{\gamma}{4} + \frac{5}{16}} P_n^{(y,x)}(-\sin \alpha), \quad (66)$$

$$G_n(\alpha) = \frac{C_n}{M + E_n} \left\{ \left[(1 - \sin \alpha)^q (1 + \sin \alpha)^p P_n^{(y,x)}(-\sin \alpha) \left(\frac{-q \cos \alpha}{(1 - \sin \alpha)} + \frac{p \cos \alpha}{(1 + \sin \alpha)} \right) \right] \right. \\ \left. + \frac{C_n}{M + E_n} \left\{ \frac{(1 - \sin \alpha)^q (1 + \sin \alpha)^p (n + x + y + 1)}{2} P_{n-1}^{(y+1,x+1)}(-\sin \alpha) \right\} \right\}. \quad (67)$$

and C_n is the normalization constant. Therefore, Eqs. (66) and (67) give the spinor wave function for the Scarf-like potential $V_2(\alpha)$ in the Dirac theory with equally mixed potentials with the same sign.

2. Solutions of the negative energy subspace of the Dirac equation in 1 + 1 dimension for $V_2(\alpha)$

This potential has the energy eigenvalues, the lower and upper component spinors, respectively, as:

$$E_n = - \left[n^2 + n \left(A' + \gamma' + \frac{5}{4} \right) + \gamma' + \frac{1}{2} + M^2 \right]^{\frac{1}{2}}, \quad (68)$$

$$G_n(\alpha) = C'_n (1 - \sin \alpha)^{\frac{B'}{4} + \frac{A'}{4} + \frac{\gamma'}{4} + \frac{5}{16}} (1 + \sin \alpha)^{-\frac{B'}{4} + \frac{A'}{4} + \frac{\gamma'}{4} + \frac{5}{16}} P_n^{(y,x)}(-\sin \alpha), \quad (69)$$

and

$$F_n(\alpha) = \frac{C'_n}{M - E_n} \left\{ \left[(1 - \sin \alpha)^q (1 + \sin \alpha)^p P_n^{(y,x)}(-\sin \alpha) \left(\frac{-q \cos \alpha}{(1 - \sin \alpha)} + \frac{p \cos \alpha}{(1 + \sin \alpha)} \right) \right] \right. \\ \left. + \frac{C'_n}{M - E_n} \left\{ \frac{(1 - \sin \alpha)^q (1 + \sin \alpha)^p (n + x + y + 1)}{2} P_{n-1}^{(y+1,x+1)}(-\sin \alpha) \right\} \right\}, \quad (70)$$

where A' , B' , γ' are as obtained in Sec. III A 2 and C_n is the normalization constant.

C. S-wave solutions of the Dirac equation in 3 + 1 dimension for $V_3(r) = \tilde{A} + \tilde{B} \coth^2 r + \tilde{C} \frac{\coth r}{\sinh r}$

1. Solutions of the positive energy subspace of the Dirac equation in 3 + 1 dimension for $V_3(r)$

For S-wave solution, substituting this potential $V_3(r)$ into Eq. (24), we have

$$\frac{d^2 F(r)}{dr^2} + \left[\epsilon - B \coth^2 r - C \frac{\coth r}{\sinh r} \right] F(r) = 0, \quad (71)$$

where

$$A = (M + E)\tilde{A}, \quad B = (M + E)\tilde{B}, \quad C = (M + E)\tilde{C}, \quad \epsilon = \tilde{E} - A. \quad (72)$$

With

$$s = \cosh r, \quad (73)$$

Eq. (71) can be written as

$$\frac{d^2 F(s)}{ds^2} + \frac{s}{s^2 - 1} \frac{dF(s)}{ds} + \frac{1}{(s^2 - 1)^2} [\epsilon(s^2 - 1) - Bs^2 - Cs] F(s) = 0 \quad (74)$$

and on comparing Eq. (74) with Eq. (5), we obtain the following relations:

$$\sigma(s) = s^2 - 1, \quad \tilde{\tau}(s) = s, \quad \sigma'(s) = 2s, \quad \tilde{\sigma}(s) = \epsilon - Bs^2 - Cs. \quad (75)$$

Substituting Eq. (75) into Eq. (12), we have

$$\pi(s) = \frac{s}{2} \pm \sqrt{s^2(B + K - \epsilon + \frac{1}{4}) + Cs + \epsilon - K}. \quad (76)$$

In order to obtain K , we equate the discriminant of the square root to zero.^{32,33} We thus obtain

$$K = \epsilon - \frac{B}{2} - \frac{1}{8} + \frac{\gamma}{2}, \quad (77)$$

where

$$\gamma = \sqrt{B^2 - C^2 + \frac{B}{2} + \frac{1}{16}}. \quad (78)$$

Therefore, $\pi(s)$ is found as

$$\pi(s) = \begin{cases} \frac{s}{2} \pm \left[\left(\frac{B}{2} + \frac{\gamma}{2} + \frac{1}{8} \right) s + \frac{C}{2} \right], & \text{for } K = \epsilon - \frac{B}{2} + \frac{\gamma}{2} - \frac{1}{8} \\ \frac{s}{2} \pm \left[\left(\frac{B}{2} - \frac{\gamma}{2} + \frac{1}{8} \right) s + \frac{C}{2} \right], & \text{for } K = \epsilon - \frac{B}{2} - \frac{\gamma}{2} - \frac{1}{8}. \end{cases} \quad (79)$$

The value of π is chosen such that $\tau'(s)$ has a negative derivative. We choose the values

$$\pi(s) = \frac{s}{2} - s \left(\frac{B}{2} + \frac{\gamma}{2} + \frac{1}{8} \right) - \frac{C}{2}, \quad K = \epsilon - \frac{B}{2} + \frac{\gamma}{2} - \frac{1}{8} \quad (80)$$

then,

$$\tau'(s) = 2 - \left(B + \gamma + \frac{1}{4} \right), \quad (81)$$

substitution of Eq. (81) with $\sigma(s) = s^2 - 1$ and $\sigma''(s) = 2$ into Eq. (10) gives

$$\Lambda = n \left(B + \gamma - \frac{3}{4} \right) - n^2. \quad (82)$$

On using Eq. (80) in Eq. (11), we have

$$\Lambda = \epsilon - B + \frac{1}{4}, \quad (83)$$

equating Eq. (82) to Eq. (83) yields

$$E^2 - M^2 = n \left(B + \gamma - \frac{3}{4} \right) - n^2 + B - \frac{1}{4} + A, \quad (84)$$

from which we obtain

$$E_n = \left[n \left(B + \gamma - \frac{3}{4} \right) - n^2 + B - \frac{1}{4} + A + M^2 \right]^{\frac{1}{2}}. \quad (85)$$

Equation (85) is the energy equation for the Scarf-Grosche potential $V_3(r)$ for the Dirac theory with equally mixed potentials with the sign.

To obtain the eigenfunctions, we take the factorization in Eq. (6). Substituting $\pi(s)$ in Eq. (80) and $\sigma(s)$ in Eq. (75) into Eq. (9), we obtain

$$\phi(s) = (s^2 - 1)^{\frac{3}{8} - \frac{B}{2} - \frac{\gamma}{2}} \left(\frac{s+1}{s-1} \right)^{\frac{C}{4}}, \quad (86)$$

and substituting $\sigma(s) = s^2 - 1$ into Eq. (13), we have

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [(s^2 - 1)^n \rho(s)], \quad n = 0, 1, 2, \dots \quad (87)$$

C_n is the normalization constant.

Also from Eq. (14), we have (after substituting for $\tau(s)$ and $\sigma(s)$)

$$\rho(s) = (s^2 - 1)^{-\frac{B}{2} - \frac{\gamma}{2} - \frac{1}{8}} \left(\frac{s+1}{s-1} \right)^{\frac{C}{2}} \quad (88)$$

and the substitution of Eq. (88) into Eq. (87) yields

$$y_n(s) = (-1)^n C_n (1-s)^{-\alpha_1} (1+s)^{-\beta_1} \frac{d^n}{ds^n} [(1-s^2)^n (1-s)^{\alpha_1} (1+s)^{\beta_1}], \quad (89)$$

where

$$\alpha_1 = - \left(\frac{C}{2} + \frac{\gamma}{2} + \frac{B}{2} + \frac{1}{8} \right); \quad \beta_1 = \left(\frac{C}{2} - \frac{\gamma}{2} - \frac{B}{2} - \frac{1}{8} \right). \quad (90)$$

Equation (89) is the Jacobi polynomials and it can be written as^{71,72}

$$y_n(s) = C_n P_n^{(\alpha_1, \beta_1)}(s), \quad (91)$$

hence $\psi(s)$ can then be written as:

$$\psi(s) = C_n (s^2 - 1)^{\frac{3}{8} - \frac{\gamma}{2} - \frac{B}{2}} \left(\frac{s+1}{s-1} \right)^{\frac{C}{4}} P_n^{(\alpha_1, \beta_1)}(s). \quad (92)$$

Therefore, the upper component spinor is obtained as

$$F_n(r) = C_n \left(\frac{\cosh r + 1}{\cosh r - 1} \right)^{\frac{C}{4}} (\cosh^2 r - 1)^{\frac{3}{8} - \frac{\gamma}{2} - \frac{B}{2}} P_n^{(\alpha_1, \beta_1)}(\cosh r), \quad (93)$$

and C_n is the normalization constant.

Following the same procedure as before, we obtain the lower spinor wave function $G(r)$ as

$$G_n(r) = \frac{C_n}{M + E_n} \left\{ \left[(\cosh r + 1)^p (\cosh r - 1)^q P_n^{(\alpha_1, \beta_1)}(\cosh r) \left(\frac{p \sinh r}{(\cosh r + 1)} + \frac{q \sinh r}{(\cosh r - 1)} \right) \right] \right\} \\ + \frac{C_n}{M + E_n} \left\{ \frac{(\cosh r + 1)^p (\cosh r - 1)^q (n + \alpha_1 + \beta_1 + 1)}{2} P_{n-1}^{(\alpha_1+1, \beta_1+1)}(\cosh r) \right\}, \quad (94)$$

where $p = \frac{C}{4} + \frac{3}{8} - \frac{\gamma}{2} - \frac{B}{2}$ and $q = -\frac{C}{4} + \frac{3}{8} - \frac{\gamma}{2} - \frac{B}{2}$. Substituting Eqs. (93) and (94) into Eq. (17) or (18) will give the spinor wave function for the Scarf-Grosche potential $V_3(r)$ for the Dirac theory with equally mixed potentials with same sign.

2. S-wave Solutions of the negative energy subspace of the Dirac equation in 3 + 1 dimension for $V_3(r)$

The S-wave solutions of the negative energy subspace of the Dirac equation in 3 + 1 dimension for this potential, that is, energy eigenvalues, the lower and upper component spinors are obtained, respectively, as:

$$E_n = - \left[n \left(B' + \gamma' - \frac{3}{4} \right) - n^2 + B' - \frac{1}{4} + A' + M^2 \right]^{\frac{1}{2}}, \quad (95)$$

$$G_n(r) = C'_n \left(\frac{\cosh r + 1}{\cosh r - 1} \right)^{\frac{C'}{4}} (\cosh^2 r - 1)^{\frac{3}{8} - \frac{\gamma'}{2} - \frac{B'}{2}} P_n^{(\alpha'_1, \beta'_1)}(\cosh r), \quad (96)$$

and

$$F_n(r) = \frac{C'_n}{M - E_n} \left\{ \left[(\cosh r + 1)^p (\cosh r - 1)^q P_n^{(\alpha'_1, \beta'_1)}(\cosh r) \left(\frac{p \sinh r}{(\cosh r + 1)} + \frac{q \sinh r}{(\cosh r - 1)} \right) \right] \right\} \\ + \frac{C'_n}{M - E_n} \left\{ \frac{(\cosh r + 1)^p (\cosh r - 1)^q (n + \alpha'_1 + \beta'_1 + 1)}{2} P_{n-1}^{(\alpha'_1+1, \beta'_1+1)}(\cosh r) \right\}. \quad (97)$$

The values of the substitutions used are as follows: $A' = (M - E)\tilde{A}$, $B' = (M - E)\tilde{B}$, $C' = (M - E)\tilde{C}$, $\gamma' = \sqrt{B'^2 - C'^2 + \frac{B'}{2} + \frac{1}{16}}$, $p = \frac{C'}{4} + \frac{3}{8} - \frac{\gamma'}{2} - \frac{B'}{2}$, $q = -\frac{C'}{4} + \frac{3}{8} - \frac{\gamma'}{2} - \frac{B'}{2}$, $\alpha'_1 = -\left(\frac{C'}{2} + \frac{\gamma'}{2} + \frac{B'}{2} + \frac{1}{8}\right)$; $\beta'_1 = \left(\frac{C'}{2} - \frac{\gamma'}{2} - \frac{B'}{2} - \frac{1}{8}\right)$ and C'_n is the normalization constant.

D. S-wave solutions of the Dirac equation for $V_4(r) = \tilde{A} + \tilde{B} \tanh^2 r + \tilde{C} \frac{\tanh r}{\cosh r}$

1. Solutions of the positive energy subspace of the Dirac equation in 3 + 1 dimension for $V_4(\alpha)$

Following similar procedure as in Sec. III C 1, we have

$$\frac{d^2 F(r)}{dr^2} + \left[\epsilon - B \tanh^2 r - C \frac{\tanh r}{\cosh r} \right] F(r) = 0, \quad (98)$$

where

$$A = (M + E)\tilde{A}, \quad B = (M + E)\tilde{B}, \quad C = (M + E)\tilde{C}, \quad \epsilon = \tilde{E} - A. \quad (99)$$

With $s = \sinh r$, we have

$$\frac{d^2 F(s)}{ds^2} + \frac{s}{1+s^2} \frac{dF(s)}{ds} + \frac{1}{(1+s^2)^2} [\epsilon(1+s^2) - Bs^2 - Cs] F(s) = 0. \quad (100)$$

On applying the Nikiforov-Uvarov method, we obtain the energy eigenvalues as

$$E_n = \left[n \left(B + \gamma - \frac{3}{4} \right) - n^2 + B - \frac{1}{4} + A + M^2 \right]^{\frac{1}{2}}, \quad (101)$$

where

$$\gamma = \sqrt{B^2 + C^2 + \frac{B}{2} + \frac{1}{16}}. \quad (102)$$

Also, we have

$$y_n(s) = D_n (1+s^2)^{B+\gamma+\frac{1}{4}} \exp \left[\frac{C}{2} \tan^{-1}(s) \right] \frac{d^n}{ds^n} \left[(1+s^2)^{n-B-\gamma-\frac{1}{4}} \exp \left[-\frac{C}{2} \tan^{-1}(s) \right] \right], \quad (103)$$

Eq. (103) is the Romanovski polynomials as^{73,74}

$$y_n(s) = D_n R_n^{(\beta, \alpha)}(s). \quad (104)$$

We therefore obtain the upper spinor wave function as

$$F_n(r) = D_n (1 + \sinh^2 r)^{\gamma_1} \exp \left[-\frac{C}{2} \tan^{-1}(\sinh r) \right] R_n^{(\beta, \alpha)}(\sinh r), \quad (105)$$

and the lower spinor wave function is obtained as

$$\begin{aligned} G_n(r) &= \frac{D_n}{M + E_n} \left\{ \left[(\cosh^2 r)^{\gamma_1} \exp \left[-\frac{C}{2} \tan^{-1}(\sinh r) \right] \left(\sinh 2r \operatorname{sech}^2 r + \gamma_1 - \frac{C}{2} \operatorname{sech}^2 r \right) \right] R_n^{(\beta, \alpha)}(\sinh r) \right\} \\ &+ \frac{C_n}{M + E_n} \left\{ (\cosh^2 r)^{\gamma_1} \exp \left[-\frac{C}{2} \tan^{-1}(\sinh x) \right] n(2\alpha + n - 1) R_{n-1}^{(\beta, \alpha+1)}(\sinh r) \right\}, \end{aligned} \quad (106)$$

where $\gamma_1 = \frac{3}{8} - \frac{B}{2} - \frac{\gamma}{2}$, $\alpha = -B - \gamma + \frac{3}{4}$, $\beta = C$, and D_n is the normalization constant. Substituting Eqs. (105) and (106) into Eq. (17) or (18) gives the spinor wave function for the Scarf-Grosche potential $V_4(r)$ for the Dirac theory with equally mixed potentials.

On a careful inspection of the potential V_3 in Sec. III C, we discovered that the potential V_3 can be transformed into the potential V_4 . This can be achieved by using the following transformations in the potential V_3 (in Eq. (3)):

$$r \rightarrow r + \frac{i\pi}{2}, \quad \tilde{C} \rightarrow i\tilde{C}, \quad \sinh r \rightarrow i \cosh r, \quad \cosh r \rightarrow i \sinh r. \quad (107)$$

It is evident that the energy equation for the potential V_4 is the same as the energy equation obtained for the potential V_4 , that is, energy equations (85) and (101) are the same.

For the upper spinor $F(r)$, we make use of the above transformations (Eq. (107)) in Eq. (93), then, $F(r)$ becomes

$$F_n(r) = (-1)^{\gamma_1 + \frac{iC}{4}} D_n \left(\frac{1 + \sinh r}{1 - \sinh r} \right)^{\frac{iC}{4}} (1 + \sinh^2 r)^{\gamma_1} P_n^{(\alpha_1, \beta_1)}(i \sinh r), \quad (108)$$

where $\gamma_1 = \frac{3}{8} - \frac{\gamma}{2} - \frac{B}{2}$, $\alpha_1 = -\frac{iC}{2} - \frac{\gamma}{2} - \frac{B}{2} - \frac{1}{8}$, $\beta_1 = \frac{iC}{2} - \frac{\gamma}{2} - \frac{B}{2} - \frac{1}{8}$.

By using the following relation:⁷¹

$$\left(\frac{1 + ix}{1 - ix} \right)^{\frac{1}{2i}} = e^{\tan^{-1}(x)}, \quad (109)$$

then

$$\left(\frac{1 + \sinh r}{1 - \sinh r} \right)^{\frac{iC}{4}} = e^{-\frac{C}{2} \tan^{-1}(\sinh r)}. \quad (110)$$

Also, the relation between the Jacobi and Romanoski polynomials is given as^{73–78}

$$R_n^{\beta, \alpha}(z) = i^n P_n^{\left(\frac{\beta - i\alpha - 1}{2}, \frac{\beta + i\alpha - 1}{2}\right)}, \quad (111)$$

in this case $\frac{\beta - i\alpha - 1}{2} = \alpha_1$, $\frac{\beta + i\alpha - 1}{2} = \beta_1$, which implies that $\alpha = C$, $\beta = -\gamma - B + \frac{3}{4}$.

Therefore, using the above substitutions, Eq. (108) becomes

$$F_n(r) = D_n (1 + \sinh^2 r)^{\gamma_1} \exp \left[-\frac{C}{2} \tan^{-1}(\sinh r) \right] R_n^{(\beta, \alpha)}(\sinh r), \quad (112)$$

where D_n is some state dependent purely imaginary complex number. With the transformations in Eq. (107), the upper spinor $F_n(r)$ in Eq. (93) of the potential V_3 reproduces the same upper spinor $F_n(r)$ in Eq. (105) of the potential V_4 . Similarly, the lower spinor $G_n(r)$ in Eq. (94) of the potential V_3 reproduces the same lower spinor $G_n(r)$ in Eq. (106) of the potential V_4 .

2. Solutions of the negative energy subspace of the Dirac equation in 3 + 1 dimension for $V_4(r)$

In the similar manner, the S-wave solutions of the negative energy subspace of the Dirac equation in 3 + 1 dimension for this potential are obtained, respectively, as:

$$E_n = - \left[n \left(B' + \gamma' - \frac{3}{4} \right) - n^2 + B' - \frac{1}{4} + A' + M^2 \right]^{\frac{1}{2}}, \quad (113)$$

$$G_n(r) = C'_n (1 + \sinh^2 r)^{\gamma'_1} \exp \left[-\frac{C'}{2} \tan^{-1}(\sinh r) \right] R_n^{(\beta', \alpha')}(\sinh r), \quad (114)$$

TABLE I. The bound state energy levels of the Dirac equation with the equal scalar and vector potentials $V_1(\beta)$ or $V_2(\alpha)$ (that is, Eq. (48) or (64)), $\tilde{A} = 2$, $\tilde{B} = 1$, $M = 1$.

n	E_{n-1}
0	2.932243823
1	6.885851447
2	10.85930472
3	14.83763195
4	18.81788851
5	22.79910343

TABLE II. The bound state energy levels of the Dirac equation with the scalar and vector potential $V_1(\beta)$ or $V_2(\alpha)$ equal to the opposite sign (that is, Eqs. (61) or (68)), $\tilde{A} = 2$, $\tilde{B} = 1$, $M = 1$.

n	$E_{n,1}$
0	-1.760522376
1	-5.246931477
2	-9.159875302
3	-13.11685658
4	-17.08639835
5	-21.06121047

and

$$\begin{aligned}
 G_n(r) &= \frac{C'_n}{M - E_n} \\
 &\times \left\{ \left[(\cosh^2 r)^{\gamma'_1} \exp \left[-\frac{C'}{2} \tan^{-1}(\sinh r) \right] \left(\sinh 2r \operatorname{sech}^2 r + \gamma'_1 - \frac{C'}{2} \operatorname{sech}^2 r \right) \right] R_n^{(\beta', \alpha')}(\sinh r) \right\} \\
 &+ \frac{C'_n}{M - E_n} \left\{ (\cosh^2 r)^{\gamma'_1} \exp \left[-\frac{C'}{2} \tan^{-1}(\sinh r) \right] n(2\alpha + n - 1) R_{n-1}^{(\beta', \alpha'+1)}(\sinh r) \right\}, \quad (115)
 \end{aligned}$$

The values of the substitutions used are as follows: $A' = (M - E)\tilde{A}$, $B' = (M - E)\tilde{B}$, $C' = (M - E)\tilde{C}$, $\gamma' = \sqrt{B'^2 - C'^2 + \frac{B'}{2} + \frac{1}{16}}$, $p = \frac{C'}{4} + \frac{3}{8} - \frac{\gamma'}{2} - \frac{B'}{2}$, $q = -\frac{C'}{4} + \frac{3}{8} - \frac{\gamma'}{2} - \frac{B'}{2}$, $\alpha'_1 = -\left(\frac{C'}{2} + \frac{\gamma'}{2} + \frac{B'}{2} + \frac{1}{8}\right)$; $\beta'_1 = \left(\frac{C'}{2} - \frac{\gamma'}{2} - \frac{B'}{2} - \frac{1}{8}\right)$ and C'_n is the normalization constant.

In addition, using the transformations in Eq. (107) in Sec. III C 2, the energy equation (113), the lower and upper spinors $G_n(r)$ (114) and $F_n(r)$ (115) are reproduced from Eqs. (95), (96), and (97).

IV. NUMERICAL RESULTS

The energy eigenvalues for the Dirac equation obtained are the same when the scalar and vector potentials are equal for:

- (i) potentials (1) and (2), see Eqs. (48) and (64),
- (ii) potentials (3) and (4), see Eqs. (61) and (68).

In the same way, the energy eigenvalues for the Dirac equation obtained are the same when the scalar and vector potentials are equal with opposite sign for:

- (i) potentials (1) and (2), see Eqs. (85) and (101),
- (ii) potentials (3) and (4), see Eqs. (95) and (113).

TABLE III. The bound state energy levels of the Dirac equation with the equal scalar and vector potential $V_3(r)$ or $V_4(r)$ (that is, Eq. (85) or (101)), $\tilde{A} = 3$, $\tilde{B} = 2$, $\tilde{C} = 1$, $M = 20$.

n	$E_{n,-1}$
0	24.99444376
1	28.69690591
2	32.36531717
3	36.00627556
4	39.62470990
5	43.22439222

TABLE IV. The bound state energy levels of the Dirac equation with the scalar and vector potential $V_3(r)$ or $V_4(r)$ equal to the opposite sign (that is, Eq. (95) or (113)), $\tilde{A} = 5$, $\tilde{B} = 2$, $\tilde{C} = 1$, $M = 2$.

n	$E_{n,1}$
0	-1.977225575
1	-1.881419081
2	-1.713971401
3	-1.508798204
4	-1.281118082
5	-1.038879559

The energy levels $E_{n,-1}(S(r) = V(r))$ or $E_{n,1}(S(r) = -V(r))$ for these sets of potentials are implicitly defined by eigenvalue equations (48) or (64) and (61) or (68) and (85) or (101) and (95) or (113). These eigenvalue equations are complicated transcendental equations, each of these equations has many solutions for the energy levels $E_{n,1}$. For these solutions, we choose suitable one as an acceptable solution.

To calculate the bound state energy levels of the Dirac equation with the equal scalar and vector potentials $V_1(\beta)$ or $V_2(\alpha)$ (i.e., Eqs. (48) or (64)), we take a set of parameters $\tilde{A} = 2$, $\tilde{B} = 1$, $M = 1$ to give the numerical results as shown in Table I.

We take a set of parameters $\tilde{A} = 2$, $\tilde{B} = 1$, $M = 1$ to give the numerical results for the bound state energy levels of the Dirac equation with $S(x) = -V(x)$ for potential $V_1(\beta)$ or $V_2(\alpha)$ (that is, Eq. (61) or (68)) as shown in Table II.

In the similar manner, the numerical results of the bound state energy levels of the Dirac equation with the equal scalar and vector potentials $V_3(r)$ or $V_4(r)$ (i.e., Eqs. (85) or (101)) are obtained as shown in Table III. The set of parameters used are: $\tilde{A} = 3$, $\tilde{B} = 2$, $\tilde{C} = 1$, $M = 20$.

Again, we take a set of parameters $\tilde{A} = 1$, $\tilde{B} = 1$, $\tilde{C} = 3$, $M = 2$ to give the numerical results for the bound state energy levels of the Dirac equation with $S(r) = -V(r)$ for potential $V_3(r)$ or $V_4(r)$ (that is, Eq. (95) or (113)) as shown in Table IV.

V. CONCLUSION

In conclusion, we have discussed the solutions of the Dirac equation with equal scalar and vector Scarf-Grosche potentials (four sets) in $3 + 1$ and $1 + 1$ space-time dimension by using the Nikiforov-Uvarov method. The bound state solutions of the Dirac equation with equally mixed Scarf-Grosche potentials have been obtained. The two component spinor eigenfunctions and energy eigenvalues for the s -wave ($\kappa = -1$ or 1) have been obtained using the Nikiforov-Uvarov method.

It is pertinent to note that the results obtained in each of these potentials are consistent with the available results in the literature, where path integrals technique has been used (see Ref. 52). The positive and negative energy solutions subspace have been obtained for these sets of potentials.

The numerical results for the bound state solutions of these sets of potentials are obtained for $n = 0$ to $n = 5$, these are shown in Tables I–IV.

Furthermore, it is discovered that the potential V_3 can be transformed into the potential V_4 by using appropriate transformations. Therefore, with these appropriate transformations, the energy equations, the lower and upper component spinors of the potential V_4 can be deduced from the solutions of the Dirac equation of the potential V_3 .

These kinds of potential (and their periodic extensions) can be used to describe atomic interactions in quantum theory (scattering and bound states), solid-state physics, and so on.^{52,54,55} This method can also be used to find the bound state solutions for the Dirac equation in $3 + 1$ space-time dimension, or Schrödinger, or Klein-Gordon equation as well (for all ℓ (orbital angular momentum)). Scattering state solutions can also be obtained via this method too.

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